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26—THE "HANDLE" OF CLOTH AS A MEASURABLE QUANTITY

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ABSTRACT

In judging the feel or "handle" of a material, use is made of such sensations as stiffness or limpness, hardness or softness, and roughness or smoothness. It is desirable to devise physical tests that analyse and reflect the sensations felt and assign numerical values to the measurements. The present paper describes tests that reflect the first groups of sensations, namely, stiffness and hardness, whilst the sensation that is experienced in stroking a material, obviously connected with frictional properties, will be dealt with in another paper.

An instrument is described on which it is possible to measure the angle through which a specimen of cloth droops when a definite length is held out over an edge. By means of a mathematical formula that is fully developed in an appendix to the paper this angle is converted into a term called the *bending length* of the material. This quantity may be defined as the length of fabric that will bend under its own weight to a definite extent. It is strictly a measure of the draping quality of a fabric. The stiffer the material, the longer is the "bending length."

From the "bending length" and the weight of the material, a simple calculation gives a quantity called the *flexural rigidity* which measures the resistance to bending or the stiffness that would be appreciated by the fingers. These two quantities should be measured both along the weft and along the warp in woven fabrics, but it is unnecessary to apply the test in any diagonal direction. The stiffness of the fabric as a whole is completely governed by the warp-way and weft-way stiffnesses, and these separate quantities may be converted into a single quantity for stiffness in any direction.

Another property that is sensed when a fabric is grasped is thickness, but this depends on the amount by which the material is squeezed, so that the sensation combines that of thickness and hardness. The measurement of thickness is discussed in detail, for it is not so simple as it might appear. Having standardised the method, however, it is easy to measure hardness by determining thickness under different pressures.

It is often desirable to compare the stiffness of materials of different thicknesses, for example, to compare a cloth before and after a process like calendering. For this purpose the above flexural rigidity may be converted into a quantity called the *bending modulus* that takes account of thickness. Flexural rigidity itself is highly dependent on thickness; in fact, doubling the thickness increases the flexural rigidity eightfold. The bending modulus on the other hand, is in a sense a measure of the intrinsic stiffness of the material. Generally speaking, it reflects the degree of compactness, or of the adhesion between fibres and threads—the difference between what is described as a "full" handle and "papery."

The standard procedure for determining bending length may be varied with regard to the shape and dimensions of the test specimen for very stiff or very limp materials, and the alternative calculations are demonstrated. A different but more laborious method for measuring flexural rigidity is also described in connection with a study of the effect of humidity on the stiffness of organdie.

CONTENTS

I—INTRODUCTION

The quality of the finish of a cotton fabric is a matter that concerns its appeal to the judgment of the buyer on the evidence of his senses. The judgment depends on time and place, on seasons, fashions, and personal and racial predilections, so that it would be futile to pretend to replace expert or aesthetic appreciation by the numerical result of a physical test. The evidence of the senses, on the other hand, depends on the physical properties of the material, so that physical measurements can be of great value in providing data upon which to exercise judgment. Thus the personal judgment of strength is now largely replaced by the numerical result of a physical test, and recent work at this Institute has shown that the visual judgment of lustre and whiteness may advantageously be augmented by physical measurement.

So far, however, those qualities of a fabric included under the name "handle" have not been discussed in terms of ultimate physical properties; the work described in the present paper was initiated in order to remedy this deficiency.* As a result, personal judgment of those aspects of "handle" that can be included under the generic name "stiffness" can now be reinforced by simple physical measurements. The aspect that involves the response to stroking, however, is not here considered; this depends on frictional properties and it will be discussed in a separate communication.

The various quantities that may be used as measures of the stiffness of a fabric are enumerated below.

(1) The Bending Length, *c*.

The way in which a fabric drapes or hangs depends largely on its stiffness, i.e. its resistance to bending, and on its weight. The most important measurement described in this paper is the determination of the ratio of these two quantities expressed in suitable units, and this ratio, or for convenience its cube root, may therefore be regarded as a quantitative measure of the property on which the "hang" of the fabric depends. The cube root of the ratio is conveniently termed the "bending length," for it measures the length of fabric that will bend under its own weight to a definite extent. The stiffer the fabric, the greater is the length necessary to ensure sufficient bending, so that a high value of c corresponds to a stiff fabric, and *vice versa*.

*The present method has been in use since early in 1927. Meanwhile, three papers have been published dealing with the measurement of flexural rigidity by the deflection of a cantilever. The mathematical formulations and practical methods are, however, substantially different and not as fully worked out for application to textiles.

The actual determination of bending length is readily done with the aid of a simple testing instrument, and it is suggested that this measurement could usefully be adopted as a routine test for cloths.

(2) The Flexural Rigidity, *G*.

While the bending length is the measure of stiffness that determines the draping qualities of the fabric, the flexural rigidity is a measure of the stiffness as appreciated by the fingers. It is, in fact, the resistance to bending mentioned in (1) above—the couple on either end of a strip of unit width bent into unit curvature, that is, the pair of forces acting in opposite directions that would be appreciated as a pressure on the skin if such a bent strip were held between the finger and thumb.

The evaluation of flexural rigidity is extremely simple when the bending length has been measured. It was pointed out above that the bending length is the cube root of the ratio, resistance to bending divided by the weight per unit area, hence the only additional measurements required are the weight and area of the specimen.

For routine tests of stiffness the measurements mentioned above are sufficient. For a more thorough study, however, and for special purposes, the quantities mentioned below may be determined.

(3) The Thickness, *d*.

All the succeeding quantities depend on the measurement of the thickness of the fabric, and this property is also of interest on its own account, since it is appreciated in handling the material. The measurement of thickness is therefore discussed in detail.

(4) The Hardness, or Resistance to Compression, *H*.

The thickness of a fabric depends on the pressure applied to it, and the relation between these quantities is a measure of the hardness of the material. In order to obtain a numerical value, the thickness is measured under two definite pressures, and the ratio of the difference of pressure to the difference of thickness is used as a measure of the hardness.

(5) The Bending Modulus, *q*.

The flexural rigidity described in (2) above is highly dependent on the thickness of the specimen, for it takes more force to bend a thick strip than a thin one; in fact, doubling the thickness increases the flexural rigidity eightfold. When the thickness of the fabric is known, however, it is possible to calculate a quantity that is independent of the dimensions of the strip. For a strip of metal or other uniform (homogeneous and isotropic) material this quantity expresses the specific resistance of the material to bending and is definitely related to the resistance to extension. For a structured material like a fabric, however, the modulus so calculated has not quite the same meaning, but it may still be used to compare the stiffness of the material or weave in fabrics of different thickness. In cotton cloths it may be regarded as a measure of the compactness, and is mainly dependent on the degree of adhesion of the fibres and threads.

(6) The Compression Modulus, *h*.

The bending modulus q may be obtained as above for the two directions parallel to the surface of the fabric. From the hardness H may be evaluated the compression modulus h, which is a similar quantity for the direction normal to the fabric surface. This quantity is also a measure of the compactness of the material.

(7) The Density, ρ.

The density of the fabric is obtained by dividing the weight in grams per sq. cm. by the thickness in cm. It represents a third measure of compactness, but is more influenced by the proportion of space left between the hairs.

(8) The Extensibility, q'.

The resistance to extension of a cloth is a property that affects the personal judgment of handle. The extensibility is expressed by Young's Modulus, obtained from an autographic strip test.

The aforementioned quantities are discussed and their measurement described in the second section of this paper, which also includes a discussion of the actual values obtained from a number of fabrics. Many of these quantities depend on the primary measurement of bending length, so that the determination of this quantity is of the first importance. The method described for this measurement is available for a wide range of fabrics, but for exceptionally stiff or limp materials, or for special purposes, other methods may be desirable. Accordingly, some alternative methods are discussed in the third section of the paper, and as an example of one of them there is described an investigation of the effect of humidity on the stiffness of organdie. The formulæ used for the evaluation of the quantities are given in an appendix, which also outlines the mathematical basis of the test.

II—THE MEASUREMENT OF STIFFNESS

(1) The Bending Length, c.

The method adopted for measuring the bending length is a refinement of the common practice of judging the stiffness of a fabric by the overhang of one or more folds. A photograph of the instrument used for this purpose (the flexometer) is reproduced in Fig. 1a. The platform A is hinged at O and is on its upper surface graduated in centimetres from that point. Fixed to it at an angle of 25° are two equal pointers B, the upper edges of which pass through O when produced. At the outer end of the platform a friction piece bears on the inner surface of a thick circular arc C, of which O is the centre; the front edge of the arc is graduated in degrees from the point level with O. The pillar is surmounted by a horizontal plane D, the front edge of which is a straight line passing through O. The removable weight F fits neatly into a bed made by side pieces, the front edge being practically coincident with that of the plane when there is no specimen between. To avoid draughts, the instrument is used in a glass case with right side and top removed, and with a black or other contrasting background.

Specimens are cut from the fabric to be tested with a razor and template, accurately 6 in. by 1 in. The edges of the template are made to follow the length-wise threads as closely as possible, and a clean cut is made that is not afterwards trimmed. It is desirable that samples should have been handled as little as possible, for bending and folding alter the stiffness and produce lines of crease or weakness that make the result meaningless. Unfilled cloth may usually be ironed under a damp cloth without appreciably affecting the stiffness of the uncreased portions. Many fabrics tend to curl and twist when cut into small specimens, and this affects both the regularity and the physical meaning of the test figures. Alternative methods are discussed in Section III to meet special difficulties of this kind.

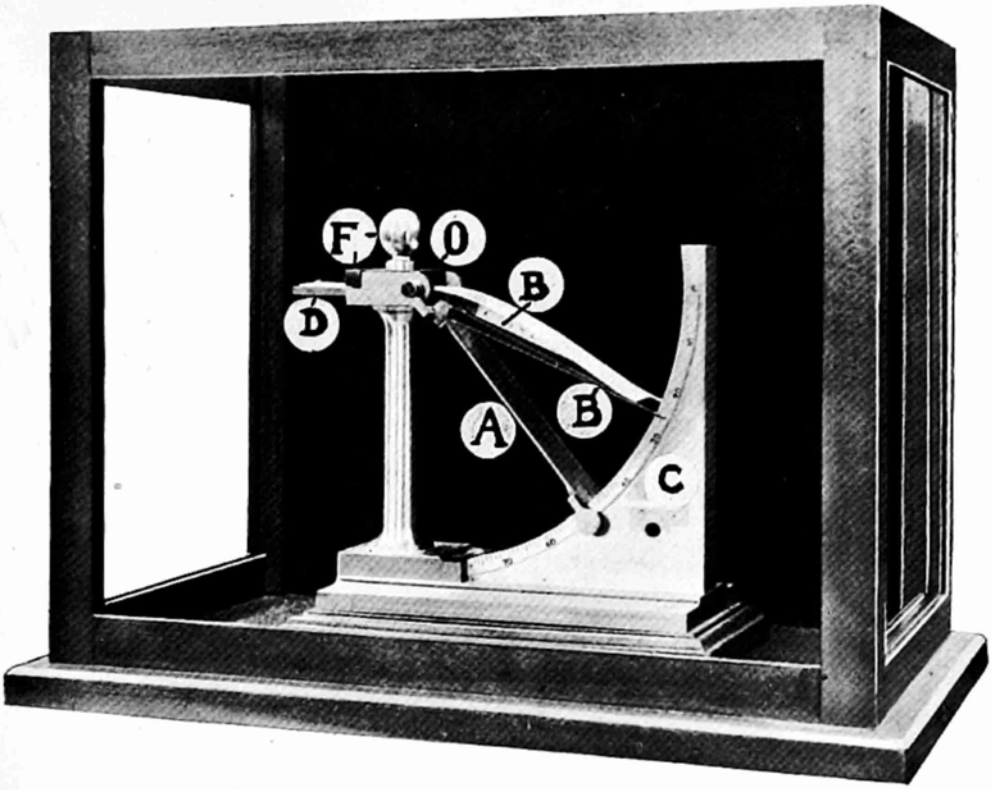


FIG. 1a. Original Model of Flexometer.

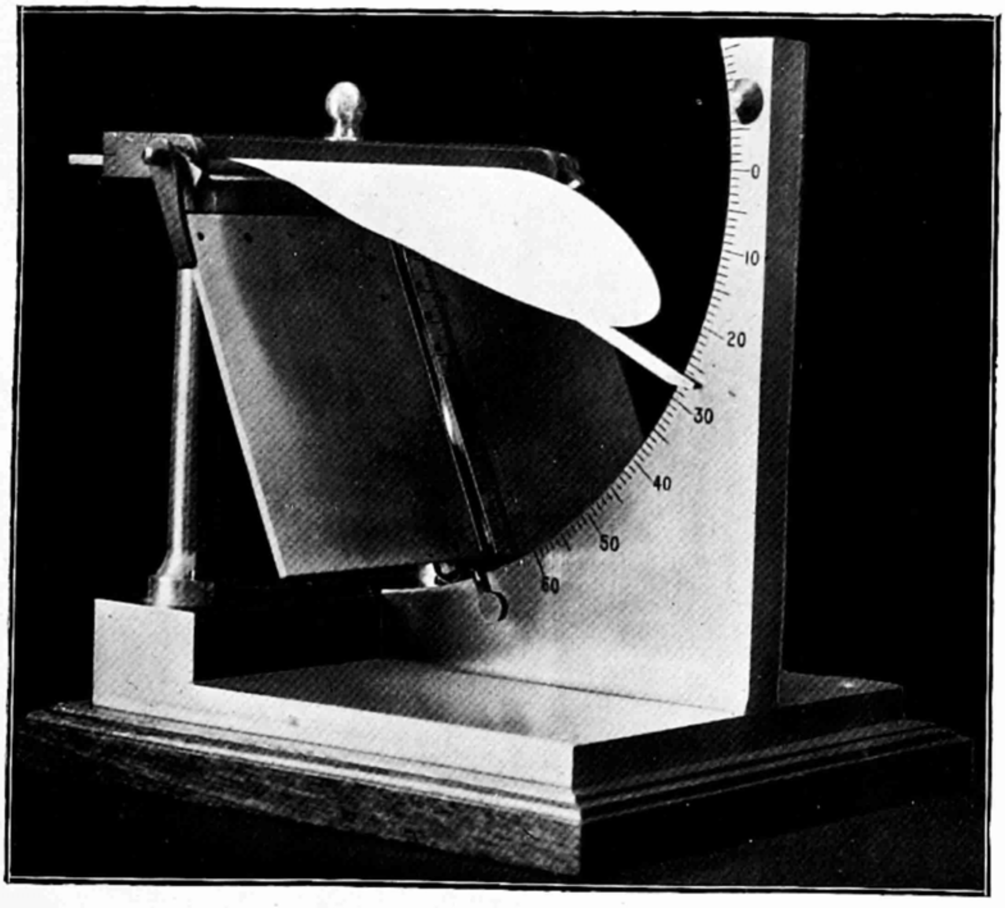


FIG. 1b. Recent model for wide specimens (see Section III).

With the platform horizontal, the specimen is laid on it and the plane D, the front end being adjusted to one of the graduations, say, 10 cm. for a stiff material. The weight F is placed in position, and the platform depressed to leave the strip overhanging. The upper edges of the pointers are then brought into line with the end of the specimen, and the reading is noted. If the end twists at all, the pointers are aligned with the midpoint. The specimen is then removed from the instrument and replaced with the other side up, when another observation is made. The two angles may differ owing to the weave or to a curl in the strip, and the mean angle is used to obtain the bending length by reference to Appendix A. The observations are then repeated on the same specimen with the ends reversed, so that two values of the bending length are obtained on any one specimen from four readings of angular deflection. It may sometimes be useful to convert the single observations to bending lengths in order to show the difference between one side and another in a fabric such as a warp-faced satin, but more often the difference is due to a curl or kink, so that the apparent difference in bending length is usually fictitious. The mean result generally depends little on the stage at which the angle is converted, for the relation is nearly linear over the useful range. For routine, it is quicker to average the four angles from each specimen before converting to bending length. The figures so calculated (one for each specimen) can then be used to judge the variability and can be related to the separate weights of the specimens.

The number of specimens necessary to give a representative value for a fent depends on the regularity of the results and the degree of accuracy demanded. Five in each direction, warp and weft, is a reasonable number. The length overhanging should be chosen to give a deflection of 20°–30°; or more for flimsy materials, since very short lengths also mean increased errors of observation. To get higher accuracy from a limited number of specimens, however, it may sometimes be useful to repeat the test on each with two or three different lengths of overhang, as the maximum bending then occurs at different parts of the strip, but if the material is available it is better to cut sufficient specimens, and to take them over as wide a range of position as possible, to get the best sampling.

An illustration of the test is afforded by the following measurements made on six warp-way specimens cut from a fent of mercerised "Tarantulle," tested at 68% R.H. with 10 cm. overhang.

Table I

Speci-men	Angles				c in Centimetres		
	End 1		End 2		End 1	End 2	Mean of Ends 1 and 2
	a	b	a	b			
1	27·9	32·0	27·3	33·5	5·943	5·902	5·923
2	21·7	27·9	25·4	29·3	6·427	6·180	6·303
3	28·7	32·2	26·1	32·2	5·902	5·984	5·943
4	27·0	31·1	31·5	34·2	6·026	5·702	5·864
5	28·5	31·3	26·9	33·0	5·943	5·943	5·943
6	26·8	31·4	29·1	29·8	6·012	5·984	5·998
	Mean angle 29·39°, c = 5·998				Mean value of c = 5·996		

In the above example the range of c on the six specimens is 0·439, corresponding to a standard deviation of 0·17 and a probable error of the mean of a five-specimen test (as recommended) of 0·08.

Between dropping the platform and taking the reading the procedure should be neither hurried nor unduly delayed. There is a slight settling down of the hanging end, depending on the material. Thus observations at 70% relative humidity on the above material (10 cm. overhang) gave an immediate reading of 26°, 28° at 30 seconds, 29° in 90 seconds, 36° after 2 hours, and 47° after 2 days. The result of the dynamical test (described in Section III) corresponds to an instantaneous reading of 23°. This time effect generally increases with the degree of bending, the softness of the material, the length used, and the humidity, but no attempt has been made to establish quantitative relations. No noticeable variation occurs when a leisurely routine is adopted, allowing about 10 seconds of free hanging (there is no need to time it). This procedure may slightly magnify the differences between stiff and soft cloths, but only in the way that they would be magnified in judging them by appearance or feel. It is most important that a swift downward swing should be avoided, for fabric will not recover completely from over-bending.

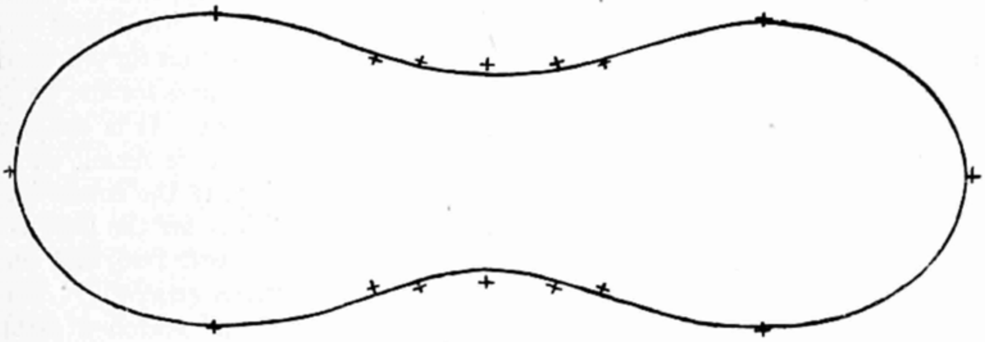
Like all textile measurements, the test for bending length is preferably done in an atmosphere of controlled humidity, failing which the hygrometer reading should be noted. In order to bend a fabric, one must bend and stretch the hairs as individuals, at least in so far as the structure binds them into a compact mass. According to quantitative relations given in other papers, the resistance both to bending and stretching decreases materially with the humidity, and consequently the stiffness of cloth must do the same. This is an important practical aspect of the question, for the moisture content can vary over a wide range without producing a definite feeling of dryness or moistness. A sample that should have a firm handle will be judged more favourably when dry than when moist, and *vice versa*.

The swelling caused by moisture absorption also enters into the question, making a compact cloth thicker and partly counteracting the tendency to "fall" or become limp. Internal slip may be minimised by the swelling and increased frictional grip of the hairs. In flimsy fabrics the bending of the hairs accounts for most of the stiffness, while in hard fabrics the stretching has more effect. The net result on the stiffness test of changes in humidity will therefore be complex, and will vary with the fabric. In a rough and general way, the effect is determined by the known relations of single hairs, but no universally valid correction could be given to reduce readings to their equivalent at a standard humidity. It is advisable to condition and test specimens in an atmosphere at or near 70% R.H., or some other constant humidity.

If the humidity effect is of special importance, the quantitative relations must be obtained by special tests, such as those described on organdie in Section III.

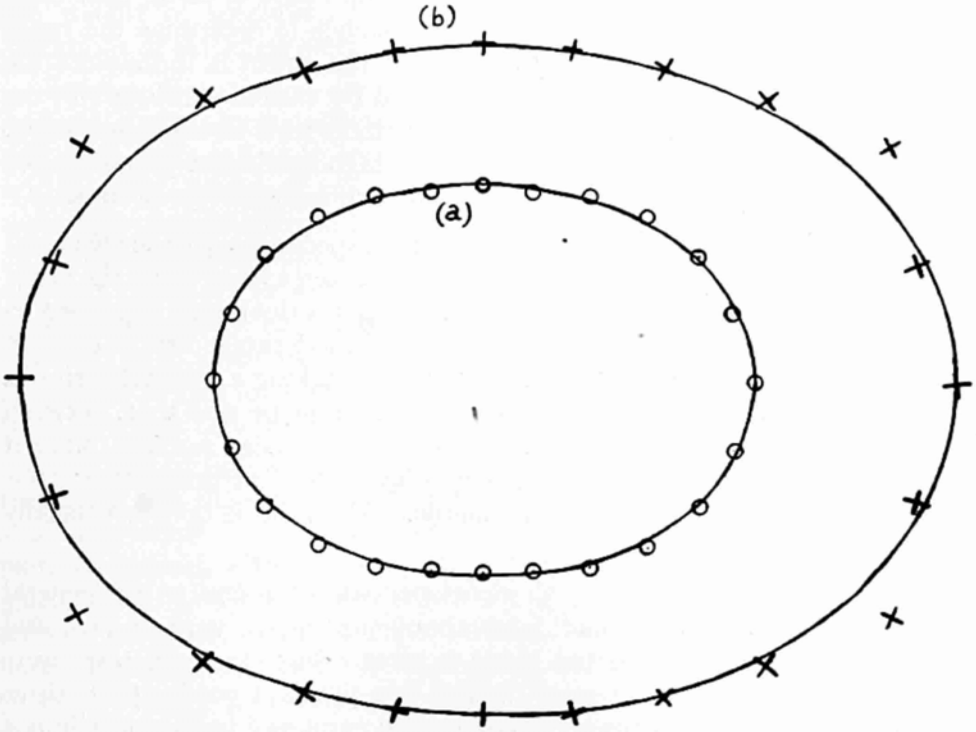
A woven fabric has different properties in the warp and weft direction; the threads generally differ in count, twist, number, crimp, and sizing, so there is no reason to expect a relation between the bending lengths measured in the two directions. Most fabrics will hang or drape quite differently according to the direction along which they are supported, and the two measurements are separately significant in describing the stiffness.

The stiffness of a fabric in a direction oblique to warp and weft depends on the values obtained in these two directions, and a formula has been derived that enables the value for any direction to be obtained when the values in the warp and weft directions are known. This formula has been confirmed experimentally. Template specimens were cut from a finished lawn at various angles to the warp, and these were tested in the usual way. In Fig. 2, the mean values of the flexural rigidity, G, in each direction are plotted, and the theoretical curve is shown fitted to them. For this material the two directions differ more than usual, and this ensured a good test of the form of the relation. In Fig. 3 the values of c obtained with a satin drill at intervals



$$\text{Curve } 1/G = \left(\frac{\cos^2\theta}{\sqrt{G_1}} + \frac{\sin^2\theta}{\sqrt{G_2}}\right)^2$$

FIG. 2: Plot of G measured on Lawn



$$c = c_1\,[1-(1-k^2)\,\sin^2\alpha]^{-\frac{3}{2}}$$

FIG. 3

Plot of c measured on Satin drill. (*a*) Bleached. (*b*) Starched and ironed by hand.

of 15° are plotted along with the theoretical curve deduced from the usual measurements in warp and weft directions. It is evident that these two measurements determine the stiffness in all directions, and hence for testing purposes there is no need to make measurements on specimens cut obliquely.

From this formula for the stiffness in any direction there can be deduced another for the stiffness averaged over all directions; this is the value that should be used when it is desired to specify the stiffness of the fabric as a whole by a single figure. The necessary formula is given in the appendix to this paper; actually the geometric mean of the values for the two directions is a close approximation to the value given by the formula, so that for practical purposes it is sufficient to multiply together the separate figures for the warp and weft directions, and take the square root of the result. It is simpler and generally sufficient to use this average figure when, as is usual, warp and weft values do not differ greatly, but it is easy to apply the correction of the formula if it should be considered necessary. Thus for the finished lawn mentioned above, the values of c were—Warp 3·10, weft 1·86, and the geometric mean 2·40 would be used to describe the stiffness generally. The exact formula would give the value 2·47. Again, for the starched satin drill, the formula value of c differs only by 0·8% from the geometric mean of the values for the warp and weft directions.

It has been suggested that five specimens should be cut in each direction to obtain a representative value of the bending length, but no invariable rule can be laid down on this point, for the number depends on the purpose of the test and on the regularity of the material. If it is merely desired to know what figure might be expected from a certain kind of cloth, then three tests on each of a number of samples should suffice to determine the range of reasonable variation in either direction, If the object is to measure the extent to which the finish of a cloth is affected by various lubricants in the size or by various steeping processes, then many more tests would be necessary. In most comparisons there is little point in high statistical precision, for cloth cannot be reproduced very exactly, either in manufacture or finish.

The variability of the stiffness of 6 in. by 1 in. specimens may be measured by the standard deviation of the bending length, taken as 0·43 times the range of 5 (the difference between the greatest and least value), and expressed as a percentage of the mean. On small fents of moderately firm fabrics in undisturbed condition, this is usually about 5%, making a probable error of the mean of five values some 1–2 per cent. It might be said that, between similar fabrics, a difference in c of less than 1% can mean nothing, since it could not be avoided by the most painstaking control of production; 10% represents an appreciable difference in handle. The value is therefore usually given to three figures.

The standard deviation of 5% mentioned above refers to the general experience of variability in small fents about one square yard or less, and more variation may be expected along a piece. For example, tests were made on a 15-yard length of satin drill at five points 3 yards apart, three warp and three weft specimens being cut at each point and tested for stiffness, weight, and thickness (as described below). A certain tendency was found in the warp for neighbouring specimens to be similar, but the variations in the weft were quite random. The values at the two ends (six specimens to each value) were, for the warp 2·85 cm. and 3·10 cm., for the weft 2·19 cm.

and 2·20 cm. The former is a significant difference and illustrates the necessity for wide sampling, and the futility of numerous tests on a small fent.

The bending length, c, is not the only quantity by which the "hang" of a fabric might be expressed. In the mathematical analysis, the formulæ give immediately the quantity $S = c^3$, the ratio of flexural rigidity to weight, and this would have been used had it been otherwise suitable. It gives, however, a very skew distribution which is a great inconvenience in interpreting results. Attempts were made to judge the differences between fabrics from which it appeared that ratios rather then differences of S were equally appreciable over the range from soft to stiff fabrics. (This appears to be fairly generally true of sensory perceptions). For all the earlier work the quantity log S was used, and proved most convenient in the reduction and analysis of observations. On this score, the bending length c is a little inferior to log S, but much better than S. It gives, however, a neater expression, more easy to visualise than log S. For the same angular deflection the length of the overhanging strip is simply proportional to the cube root of S, i.e. to c. This quantity has been used as the prime measure of resistance to bending, both by Hummel and Morton[1] and by Petersen and Dantzig,[4] so that some possible future confusion has been avoided by adopting it for the present work.

The measure of variability also depends on the quantity used to express the observations. In the warp-way specimens of the satin drill (above), the variability of S or c^3 was 17%, of c 5·4%, while that of the weight was only 1·2%, of thickness 1·7 per cent. It was found that the bending length and the weight of specimens were well correlated (coefficient 0·7), but the variations of thickness were more localised and random.

Sources of variation are numerous in the production of a nominally uniform cloth. In manufacturing there are factors such as counts of warp and weft, picks and ends, weight of size, and warp tension, and in bleaching and finishing, factors of concentration, reaction and temperature of liquors, non-uniform circulation in baths, and the action of machines. These result as far as stiffness is concerned, mainly in variations of weight, compactness, and wax content. In well controlled production they probably do not occur to a much greater extent between different pieces than within the same piece, so that when batches or pieces vary significantly as a whole, it is reasonable to look for a definite cause; this is one of the main routine applications to which this test is suited.

Two sets of figures are available on such points, one illustrating normal regularity and the other an evident irregularity ascribable to a definite and avoidable cause. The former concerned fents from three pieces of a given mark (long-cloth with 6% starch filling), which gave values of c—Warp-way 3·020, 3·119, 3·055; weft-way 2·410, 2·529, 2·438. The other set concerned a fine calendered muslin in which stiffness was a desirable quality, and in a batch of which some pieces finished quite soft for no known reason. Differences of over 25% in warp and weft directions were closely correlated with differences of wax content, and both with thickness but not with weight. The relations are shown in Fig. 4. On investigation, the variations were explained by hitherto unrecognised variations in the weight of size put on the warp. More size meant more fat left after desizing, and also a looser construction, both lowering the stiffness of the finished piece.

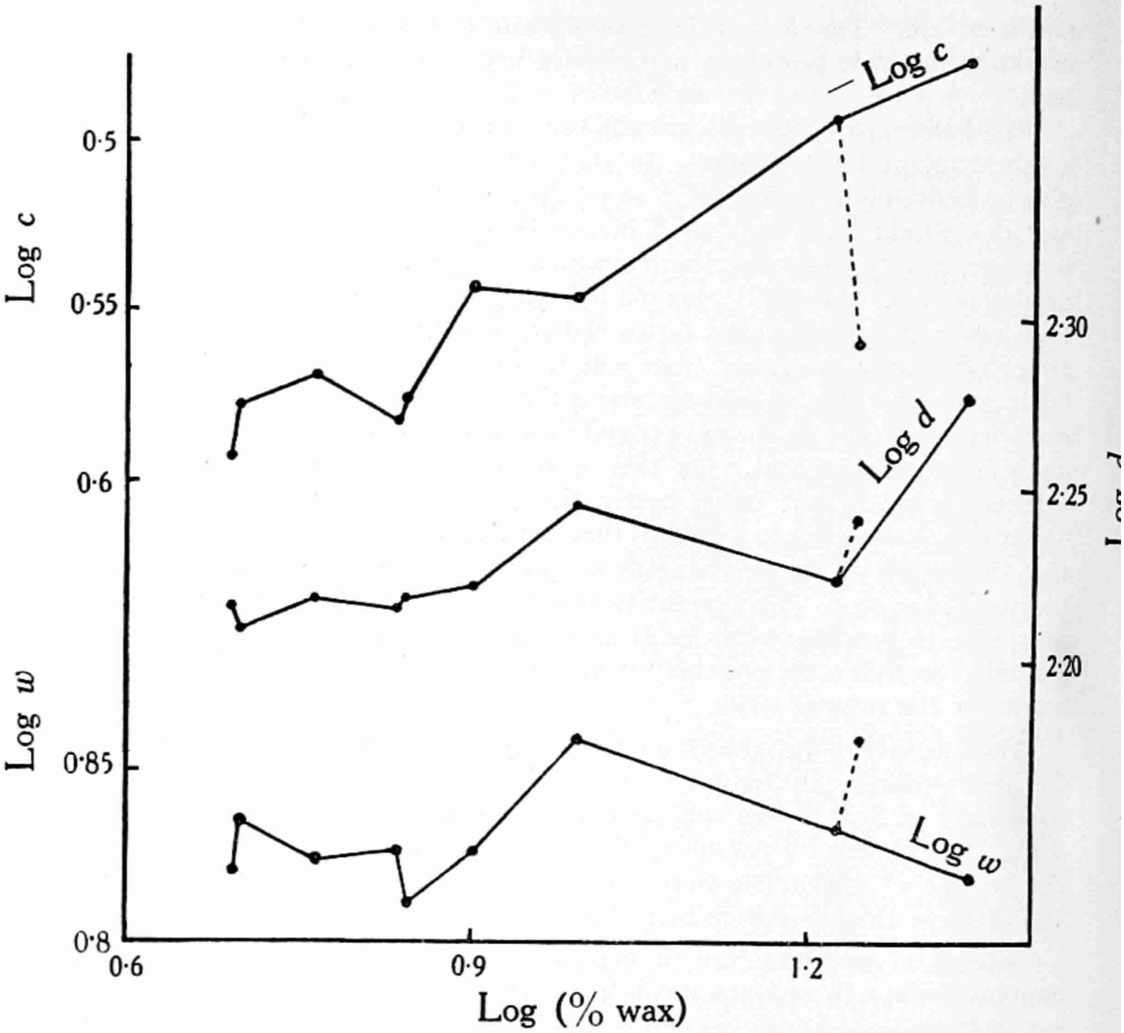


FIG. 4

Wax contents. Values from 10 pieces nominally the same.
No. 9 was very irregular and the mean values have little significance.

(2) The Flexural Rigidity, *G*.

The flexural rigidity measures the actual forces produced in bending the material; thus two fabrics may bend to the same extent under their own weight, but the heavier of the two will exert more resistance to bending, say, by the fingers, and so feel stiffer. The only additional measurement required is the weight of the fabric, whence the flexural rigidity can be calculated from the equation $G=wc^3$, where G is the flexural rigidity, w is the weight of the fabric in grams per sq. cm., and c is the bending length. The weighing of specimens is therefore recommended as a regular routine in the stiffness test. It may be done on an analytical balance, in the same atmospheric conditions as the test itself, but no great precision is necessary and other kinds of balance may legitimately be used.

As an example of the value of this quantity, the following figures for a grey calico and a lighter aircraft fabric may be compared. The calico gave values of c warp-way 2·70, weft-way 2·14; the figures for the aircraft fabric were 2·70 and 2·12. The weights were 12·1 mgm. per sq. cm. for the calico, and 6·32 mgm. per sq. cm. for the aircraft fabric, so that the flexural rigidities were warp-way 236, weft-way 118 for the former, and warp-way 123, weft-way 60 for the latter. While the fabrics were therefore identical so far as draping qualities were concerned, the calico would be much the stiffer as measured by the fingers.

(3) The Thickness, *d*

For the measurement of thickness, a micrometer dial-gauge was used, with a scale divided into units of 1/100th of a millimetre, a base plate of several square inches, and a removable upper plate or pedal. Three pedals were used, with circular plane ends of diameter 0·1, 0·25, and 1·0 inch. A small compressive force is normally applied by a spring inside the gauge, and by the weight of the pedal. This force can be diminished almost to zero by hanging a weight on the operating lever, or increased by placing disc weights on the spindle that carries the pedal.

In order to evolve a satisfactory method of measurement, observations were made on a satin drill with all the available combinations of area and pressure, which are shown below.

Table II
Areas and Pressures of Gauge Pedals

Pedal area, sq. cm.—	5·067		0·3167		0·0507	
	Load gms.	Pressure gms/cm.^{2}	Load gms.	Pressure gms/cm.^{2}	Load gms.	Pressure gms/cm.^{2}
Counterpoised ...	9·660	1·906	3·750	11·84	2·885	56·94
Unweighted ...	26·42	5·213	20·51	64·76	19·65	387·7
Weight 1... ...	40·18	7·929	34·27	108·1	33·41	659·2
Weights 1+2 ...	53·88	10·63	47·97	151·5	47·11	929·6
Weights 1+2+3	79·26	15·64	73·35	231·6	72·49	1430
Weight 3 ...	—	—	45·89	144·9	—	—

The results are given in Table III in microns (μ=10^{-4} cm.), each figure being the mean of 30 measurements on as many strips of 6×1 inches cut from a 15 yard length of cloth. The average standard deviation was 11·8, and the probable error of the mean 1·5.

Table III

Pedal	Large	Medium	Small
Counterpoised	660 ...	569 ...	453
Unweighted	584 ...	488 ...	420
Weight 1	559 ...	470 ...	404
Weights 1 + 2	544 ...	458 ...	394
Weights 1 + 2 + 3 ...	529 ...	443 ...	381

It is evident from the table that under a light pressure over a large area the thickness is very sensitive to the pressure; this is due to the fact that contact is made over comparatively little of the surface. When the protruding portions are pressed flat and the foot comes in contact with more of the surface, the variation with pressure diminishes.

Since the thickness of a cloth is variable, the measured thickness is determined by the thick or hard places, and hence the larger the pedal the greater is the chance of including thick spots, and therefore the greater the average measured thickness. For an irregular incompressible fabric, the underlying theory of the effect of the size of the tested specimen is identical with that of the effect of the length of the specimen on the breaking load; this has been discussed in an earlier paper.[5] As an example, the thickness of a starched twill under low pressure was 417 μ under a foot 0·1 in. in diameter; under a foot of 1 in. diameter it was 444 μ, agreeing with the theoretical value of 445 μ deduced from the measurements with the smaller foot.

On a uniform compressible fabric the results from different pedals would be a definite function of the pressure (total force divided by the area), but would be independent of the size of the pedal. Actual fabrics, however, are both irregular and compressible, so that most of the compressive force is taken by the thicker portions, and the effective pressure is greater on a larger pedal. As the compressibility of thick and thin places may be consistently different, the relation between thickness measurements under different areas and weights is very complex, and no simple general rules can be given.

It should be remembered further that the cloth under the pedal consists of threads that continue beyond it, so that an edge effect is to be expected. At the boundary between the compressed and free portions the threads have an upward component of tension, a resistance to compression proportionately greater on a smaller area. On the other hand, a small pedal may cause some sideways displacement in a soft cloth, an edge effect acting in the other sense. These several factors of irregularity, compressibility, and edge effect, will vary in importance with different materials, areas, and pressures. A more detailed analysis is possible, but would serve no useful purpose.

The conditions chosen for a standard measurement should be a compromise, more or less arbitrary, avoiding the undue influence of any one factor. In this work, unless otherwise stated, the measurements have been made with a foot of $\frac{1}{4}$ in. diameter without extra weighting; that is, under an area of 0·3167 sq. cm. and a load of 20·51 grams.

A method that might be mentioned for getting over the indefiniteness of thickness due to irregularity is to measure the thickness in multiple layers. The random variations compensate each other, and the average thickness per layer is usually less than that measured on single layers. This is illustrated by the following figures for the thickness of stockings, measured in four layers and singly, under the series of weights given in Table II, medium foot.

Fourfold	...	404	...	358	...	345	...	335	...	321m icrons
Single	...	445	...	395	...	380	...	370	...	355

On the other hand, the thickness measured on multiple layers of varnished electrical tape was greater than on single layers. This material is so hard that irregularities over a considerable area affect the contact at the measuring point, so effectively increasing the area of the gauge foot.

The method is adopted by the Electrical Research Association for testing insulating fabrics, but has not been used here, for it introduces another arbitrary factor and more danger of edge effects. There is no absolute definition of the "thickness" of a fabric, and it is sufficient, for comparison, to adopt the simplest reasonable conditions of measurement and maintain them invariable.

There is still room for personal variations in the manner of lowering the pedal. This has been done steadily without drop, though the American Society for Testing Materials recommends dropping the pedal from a constant height of $\frac{1}{8}$th of an inch.

(4) The Hardness, *H*.

The relation between thickness and pressure—using the medium foot—has been observed on a number of fabrics and is shown by the curves in

Fig. 5. Differences of surface and texture are reflected in the shape and slope of the curves, all of which are more or less concave upwards. Between the first two points the rapid change is characteristic of the surface rather than of the body of the cloth. To some extent this is an index of the roughness of the surface, as shown, for instance, by the curvature of the graph for gingham compared with the straightness of the graphs given by the smooth, calendered "rayonne" and lawn. The difference in these two measurements is most useful in characterising raised fabrics, and may be used as a measure of the "nap," e.g. in the flannelette.

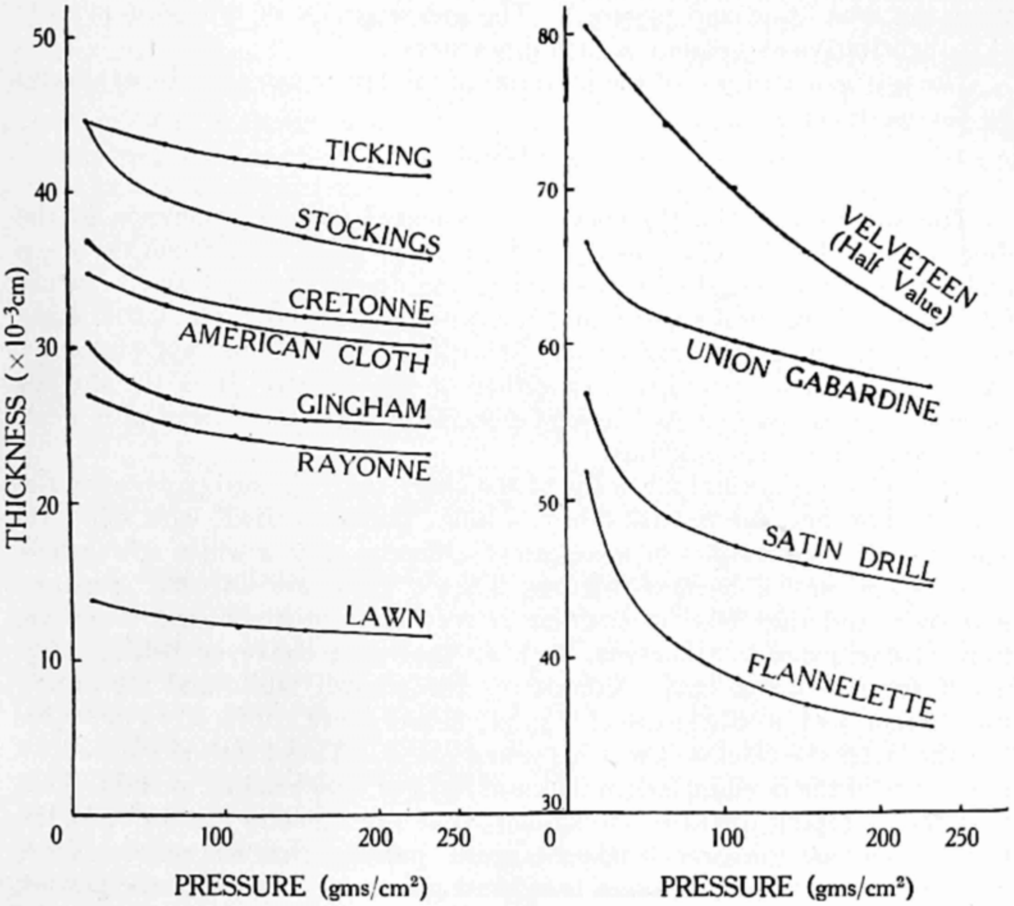


FIG. 5a FIG. 5b

At the other extreme, the fibres are pressed into contact and are themselves being compressed. It is considered that the slope between the second and fourth points is most characteristic of the texture, and this has been adopted as a measure of another factor in the "feel" of the cloth. After each thickness measurement without a weight on the pedal, another reading is taken at the same spot with weight No. 3 added. The ratio of the difference in pressure (80·14 gms./cm.²) to the difference in thickness thus observed is a measure of the "hardness," H, such as may be appreciated by pressing with a finger tip.

The pressure difference adopted here is arbitrary and may be varied for special purposes. For example, schreinered and varnished fabrics are so hard that a much greater pressure must be used in order to make a comparison sufficiently sensitive, and to test the body of the cloth rather than the surface irregularities. The same applies to sheets of continuous material, starch film, cellulose sheet, or paper, for which the smallest foot is more suitable.

(5) The Bending Modulus, *q*.

In the section on "flexural rigidity," a comparison was made between a grey calico and an aircraft fabric; the two fabrics fall under their own weight in just the same way, but the heavier fabric is appreciably more resistant to bending between the fingers. There are other differences, however, that would be described by saying that the former had a "fuller" handle, while the latter was "thin and papery." The measurement of thickness permits of a quantitative expression of such differences.

The intrinsic stiffness of the material of the fabric can then be expressed by the modulus—

$$q = 12G/d^3.$$

The weight and the thickness are measured without reference to the directions of the threads, and it has been found best to combine the warp and weft bending lengths by the average, as discussed in (1) above, before calculating the flexural rigidity and the bending modulus. The latter figure is by no means a mere theoretical abstraction; it can be appreciated by handle more easily perhaps than either of the others. It is the quality generally called "paperiness," a good descriptive name that might with much truth be given to this modulus.

The value for normal fabrics is of the order 100. A hard paper gave the value 30,110, and the nearest fabric, a tulle "parchmentised" with sulphuric acid, 17,350; as examples of mechanical stiffening only, a white schreinered satin, 4,228, and a beetled shirting, 2,343. These are all very "papery" materials, and that kind of stiffness is very easily distinguished in handle from stiffness due to thickness, such as that of a conveyor belting with $G=26{,}420$ and $q=30$ only. Comparing the aircraft fabric and the calico, the former had a thickness of 139 μ, hence $q=12\times10^6\times86 / 139^3=386$. For the latter the thickness was 324 μ and $q=59$. Thus for these two fabrics the values of the bending length, flexural rigidity, and bending modulus show that their draping qualities are similar, that the calico is the stiffer to the fingers, and that the aircraft fabric is more "papery" than the calico. These qualities could, of course, have been distinguished by handling the fabrics; the advantage of the methods here described is that definite numbers can be ascribed to these qualities, and so comparisons can be made much more easily and with less doubtful result when the differences between the materials are not so pronounced.

(6) The Compression Modulus, *h*.

Compression may be expressed as a strain, that is, the difference in thickness divided by the original thickness. The ratio of stress (difference in pressure) to strain gives a "Young's Modulus" for the material in a direction normal to the surface, which may be compared with the bending moduli obtained for the warp and weft directions. The ratio, which may be called the "compression modulus," h, depends mainly on the compactness of the material, and is in fact found to be highly correlated with the bending modulus. It is, however, more influenced by the surface irregularities of hard fabrics.

(7) The Density, ϱ.

If the weight in grams per square centimetre is divided by the thickness in centimetres, the quotient is the weight per cubic centimetre or density, denoted as is usual by ϱ. This is also a measure of compactness, but is more influenced by the proportion of space left between the threads. The bending modulus is rather a measure of the degree to which the fibres and threads are welded together mechanically.

(8) The Extensibility, q'.

In handling a cloth it is easy to appreciate its resistance to extension, and that quantity must affect the personal judgment of handle. The extensibility is expressed by the ratio of tensile stress to strain (Young's modulus), which is obtained from an autographic strip test. This test yields a curve between load and extension that is usually far from straight, the strip being more extensible at the beginning of tension, and to compare with handle and the bending test the extensibility should be judged from the initial slope.

In a rod of uniform material, Young's modulus and the bending modulus are identical. In a fabric the internal structure differentiates resistance to tension from resistance to bending, but a general relation is found between them, stiff cloths being less extensible. Thus Young's modulus was calculated from the load at 1% extension of a series of fents of white satin schreinered at various pressures. A high correlation was found between the tensile and bending moduli, and between both and thickness, as shown in Fig. 6. The resistance to tension is greatly increased by compression of the fabric in a calender, but the resistance to bending is increased very much more.

The close agreement of the two moduli for a structureless sheet has already been demonstrated for starch film.[6] A relation almost as close may be expected in heavily doped fabrics for aircraft or electrical insulation. Flexibility is an important quality of the latter when the fabric is wrapped round moving parts of an accurately machined instrument. It must bed into variations of curved surfaces, the capacity for which involves both extensibility and softness to bending. The stiffness test should be sufficient indication of both qualities.

Flexibility, in the ordinary use of the word, involves not only ease of bending but also the capacity to bend to a large degree without cracking or breaking. There is the same duality of meaning in extensibility. Again, the two quantities are fundamentally the same, the limiting amount of bending being determined by the local extension in the material of the outer surface at the greatest curvature. The measurement of ultimate extension is familiar in tensile tests. A corresponding test for ultimate bending is described in Appendix B.

Characteristics of Various Fabrics

To illustrate various applications of the stiffness test and to initiate a basis of comparison, the values obtained for the several measurable characters relating to "handle" (with the exception of the extensibility) are collected in Table IV. These are derived from measurements on a wide variety of fabrics without special selection, and they include all those examined up to July 1928, except for several special researches separately reported.

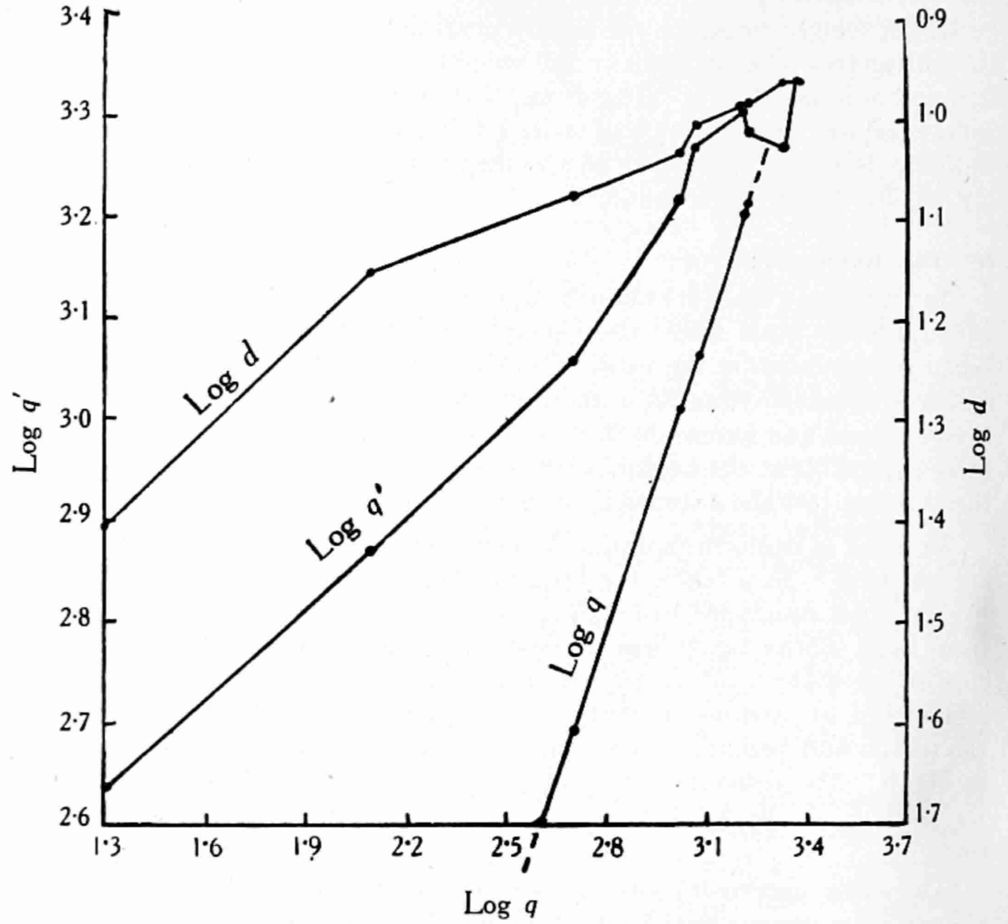


FIG. 6

Stiffness and Extensibility

Table IV

Stiffness Characteristics of Various Fabrics

The entries in the columns are—

w Weight per sq. cm., in milligrams, at 70% R.H.

d Thickness in μ (micron = 10^{-4} cm.) measured under a foot of ¼ in. diameter; load 20·5 grams.

$\varrho = w/d$, the density.

c The bending length in centimetres, from flexometer reading at 70% R.H.

$G = wc^3$, in mgm. cm., the flexural rigidity.

$q = 12G/d^3$, in kg./cm.^{3}, the bending modulus.

H The hardness, the ratio of changes in pressure and thickness in mgm./cm.^{3}

h The compression modulus = Hd, in kgm./cm.^{2}

No.	Material	w	d	ϱ	c Warp	c Weft	G	q	H	h	Notes
1	Aircraft fabric ...	6·32	139	0·45	2·70	2·12	86	386	134	1·86	L
2a	Calico	12·2	320	0·38	2·64	2·72	235	87	39	1·24	L
3	Calico	18·8	380	0·49	3·35	3·61	790	173	33	1·25	L
4	Calico with size containing tallow (a)	10·6	262	0·41	3·21	2·80	286	192	43	1·13	L.1
	(b)	10·9	263	0·41	3·29	2·86	316	209	47	1·24	L
5a	Calico, grey ...	12·1	324	0·37	2·70	2·14	169	59	31	1·02	L.2
5b	Water-boiled ...	12·0	299	0·40	2·03	1·94	93	42	32	0·96	
5c	Scoured ...	10·6	272	0·38	2·17	2·06	101	60	31	0·85	
5d	Bleached ...	10·8	263	0·45	2·55	2·03	127	84	34	0·89	B

Table IV *(continued)*

No.	Material	*w*	*d*	*ρ*	*c* Warp	*c* Weft	*G*	*q*	*H*	*h*	Notes
6a	Mull	3·53	142	0·25	1·92	1·70	21	88	73	1·04	B.3
7a	Satin	7·56	231	0·33	2·64	1·46	57	55	45	1·03	B
8a	Taffeta	8·09	242	0·34	2·09	2·06	72	61	55	1·34	B
9a	Tarantulle ...	8·66	208	0·42	2·25	2·04	85	113	75	1·56	B
10	Low weft sateen	8·70	395	0·22	2·23	1·38	47	9	22	0·86	B
2b	Calico	12·1	319	0·38	2·67	1·86	134	50	42	1·34	B
13	Harvard shirting	13·5	473	0·29	2·94	2·36	246	28	18	0·87	B
14	Long cloth ...	13·7	228	0·60	3·15	3·24	444	450	79	1·79	B
15	Calico	16·0	318	0·50	2·58	2·40	247	93	42	1·33	B
16	Twill	20·0	505	0·40	3·44	2·29	443	41	30	1·53	B
17	Satin drill ...	22·2	470	0·47	2·99	2·17	367	42	35	1·65	B
18	Mull 0·17% wax	6·62	162	0·41	3·91	2·47	199	566	76	1·23	C.4
19	Mull 0·29% wax	6·61	189	0·35	2·99	1·85	86	152	54	1·02	C.4
20	Poplin	10·5	225	0·47	2·69	1·73	105	111	71	1·60	C.5
21	Poplin	11·0	219	0·51	2·62	1·60	94	108	72	1·57	C
22a	Beetled shirting	11·7	142	0·82	3·53	3·17	438	1836	95	1·35	C.6
23a	Beetled shirting	11·7	140	0·84	3·64	3·52	537	2343	94	1·32	C
22b	Before beetling	13·3	395	0·34	2·47	2·45	198	39	36	1·41	B
23b	Before beetling	13·3	405	0·33	2·28	2·47	177	32	42	1·69	B
7b	Satin	7·42	99	0·75	5·06	2·56	347	4228	178	1·77	S
24a	Black satin ...	7·87	234	0·34	2·68	1·38	56	52	48	1·12	
24b	Black satin ...	7·88	118	0·67	4·51	2·09	228	1656	120	1·42	S
25a	Black sateen ...	20·5	486	0·42	2·22	2·15	214	22	30	1·48	
25b	Black sateen ...	21·3	356	0·60	3·33	3·02	681	182	35	1·24	S
8b	Mercerised taffeta	7·45	217	0·34	2·22	2·58	102	120	50	1·09	
6b	Mercerised mull	3·51	132	0·27	2·20	2·21	38	194	73	0·96	
6c	Organdie ...	3·39	100	0·34	5·45	3·89	331	3972	171	1·71	
9b	Linenised tulle	8·66	215	0·40	6·50	5·15	1681	2039	100	2·15	
9c	Acid treated ...	9·33	234	0·40	2·52	2·08	112	104	72	1·68	
9d	Parchmentised	13·3	154	0·86	8·00	6·78	5308	17350	123	1·90	
26a	Long cloth ...	12·2	197	0·61	3·76	2·66	383	606	99	1·95	F
26b	Filling removed	10·8	244	0·44	2·36	1·78	93	81	68	1·66	
27a	Long cloth ...	13·0	206	0·63	3·06	2·46	269	367	97	2·01	F
27b	Filling removed	12·3	242	0·51	2·37	2·14	139	118	71	1·72	
28	Printed lawn ...	5·52	132	0·42	3·10	1·86	774	404	98	1·29	F
29	Gingham (print)	9·80	284	0·35	6·15	2·71	667	350	55	1·56	F
30	Cretonne (print)	12·9	350	0·37	2·76	4·37	537	151	57	1·99	F
31	Flannelette (dyed)	15·9	420	0·38	4·81	2·04	487	79	21	0·87	F
32	Ticking	25·4	437	0·58	3·57	5·11	1985	285	52	2·28	F
33	Velveteen (dyed)	29·9	1450	0·21	2·69	1·85	331	1·3	5	0·69	F
34	Conveyor belting	104	2200	0·47	6·32	6·35	26420	30	8	1·72	F
35	Crepe rayon ...	6·13	179	0·34	1·55	2·82	56	118	65	1·16	F
36	Crepe print ...	10·2	270	0·38	1·67	1·92	59	36	69	1·87	F
37	Crepe shot ...	17·4	346	0·50	3·15	3·35	593	172	42	1·46	F
38	Worsted weft, twill	24·9	623	0·40	2·18	2·09	242	12	26	1·65	F
39	American cloth	32·5	331	0·98	3·23	2·47	733	242	49	1·63	F
40	Graph paper ...	4·83	75	0·64	6·03		1059	30110	211	1·58	F
41	Feeler steel ...	30·2	381	7·93	6·76		9354	$2{\cdot}03 \times 10^6$			

Notes

L, loom state; B, bleached; C, calendered; S, schreinered; F, normal commercial finish.

1—Each row of No. 4 gives the mean values from four fents having approximately 11% of size on the warp, with increasing amounts of tallow: (4a) mean of 2·4% of tallow to starch, (4b) mean of 11·9% tallow to starch.

2—Similar cloth at successive stages, indicated by a, b, etc., after the same numeral.

3—See also 6b and c below.

4—The extreme cases of Fig. 5.

5—Nos. 20 and 21 are similar cloths containing yarn from the same spinner, of which 20 appears to have been woven with a greater warp tension.

6—Two similar pieces, of which 23 had been given a lime boil, which resulted in a wax content of 0·12% against 0·22% in No. 22.

A range of stiffness from 1·81 to 6·35 cms. (mean value of *c*) is here recorded. Fabrics approaching the lower limit were too soft to be measured by the standard method, so were tested by the pear cantilever and triangle methods described in Section III. Later, the "hanging heart" method was developed, and is now used for material with a lower value than 2 cms. (which gives 43° deflection with a 4 cm. rectangular cantilever), especially if it tends to curl. The softest fabric that could be found was a knitted viscose, with a bending length of 0·6 cm., measured without difficulty by this method. The highest value does not approach the upper limit that can be measured by the weighted rectangle method (Section III)—in fact, there is no practical limit of stiffness for that method.

The flexural rigidity, G, ranges from 21 for the lightest, a mull of 3·53 mgm. per sq. cm., to 26,420 for the heaviest, belting of 104 mgm. per sq. cm. Other things being the same, the flexural rigidity varies with the third power of the thickness, but if any one cloth is calendered or otherwise compressed, the relation is reversed owing to the rise in specific stiffness. This quantity, q, is the best to consider when it is desired to estimate the effect of the structural factor, which is chiefly the extent to which the hairs cohere. There is also a close correlation between the compression modulus, h, and the density, ϱ, most of the values falling near the line $h=10\ \varrho/3$; the deviations can reasonably be explained by differences in openness of weave. Cotton cellulose has a density of about 1·5, but few of the materials reach one third of this. About the value 0·5 the hairs are evidently in intimate contact, for the specific stiffness increases rapidly. The greatest density attained mechanically is in the beetled and calendered shirting (0·84), where the finish has almost closed the spaces laterally.

The density and stiffness are influenced at many of the early stages of production; the raw cotton, the counts and twist of yarns, the amount and composition of the size, the number of picks and ends, and the warp tension. Thus the difference between poplins 20 and 21 was associated with lower extensibility and crimp, i.e. greater warp tension, in the stiffer cloth.

The presence of lubricating or cementing matter on the hairs has evidently a big effect. Thus the removal of wax in bleaching increases the stiffness (compare the figures for the water-boiled and bleached calico, Nos. 5b and 5d). The absence of wax is also responsible for the excessive harshness caused by mercerising after a scour. The effect of wax in impeding the development of stiffness in mechanical finishes is shown by the figures for the mulls, Nos. 18 and 19, and the beetled shirting, Nos. 22a and 23a. In the grey state, fats in the size do not appear to have any effect on the stiffness. Starch, as size or filling, has a great stiffening effect, more by cementing the hairs than by virtue of its own rigidity. Dyeing seems to have the opposite effect, the diminution of stiffness being particularly noticeable in the schreinered satins; heavy printing has an appreciable stiffening effect. The swelling in mercerisation brings the hairs into intimate contact, so that the specific stiffness is doubled in the examples given. Sulphuric acid, in the organdie and parchmentising treatments, multiplies the modulus 50 or 100 times, virtually welding the hairs together.

The stiffness of cloth is ultimately dependent on the elastic properties of the hairs, that is, on the sum of the forces necessary to bend and to stretch them. If it is assumed that half the hairs lie in the direction of the strip, then the force necessary for bending alone is $G/2$, where G is the flexual ridigity of single hairs, and the corresponding value of c will be $\sqrt[3]{G/2m}$, where m is the hair weight per centimetre of single hairs. From data formerly obtained, the value of $G/2m$ for Texas hairs at 70% R.H. is about 6 cm.[3] or $c=1{\cdot}82$. This agrees with the value for the very flimsiest cloth, the low weft sateen, in which the fibres are apparently so loosely held that they do not take any tension. At the other limit of a compact continuous sheet with elastic properties the same as those of the hair wall material, the value of q would be about 60,000 kgm./cm.[2] It is, of course, impossible to realise such a value, and a nearer approach could hardly be made than that made by the parchmentised sheet with a modulus, $q=17{,}350$. There is no relative movement of the fibres in this, and the lower modulus is sufficiently explained by the degradation of the cellulose, the two orientations of the hairs, and the imperfect uniformity of the sheet. In all other fabrics more or less air space is included, and the hairs take some tension but also yield by relative movement.

The graph paper is also mainly cellulose and is of high quality, thin, tough, and uniform, and gives a value of q half that of the Young's modulus of cotton hairs. It may be noted that the value for feeler steel agrees with the Young's modulus given in tables of physical constants.

The figures of the table yield on perusal many more interesting relations. They express facts that could be appreciated in handling the cloths, but in definite numbers and an impersonal, unified scheme of comparison. More specific investigations have been made, yielding regular curves against some variable factor of treatment, such, for example, as temperature and concentration of mercerising liquor, but the present paper is concerned with the methods of test and their meaning, not with particular problems of finishing in which it may be used.

III—ALTERNATIVE METHODS

In the standard method described in Part I, it is assumed that the specimen of fabric lies flat when unstressed, and that the measured deflection from a horizontal plane corresponds to bending under the weight of overhanging cloth. The construction of many fabrics and the strains produced in finishing often produce, however, a tendency to curl and twist. If the distortion is moderate, its effect is largely eliminated by the two readings with face up and down, but some materials twist into almost helical form. No very precise measurements can be made on such materials, but various modifications of the method have been worked out to deal with those unsuitable for the standard method, and also to increase the range of stiffness measurable.

In the form of the standard rectangular strip with an overhang of 8 cms., fabrics may be tested that give a deflection from 10° to 50°, covering the range of c from about 7 to 3 cms. The length may be varied from 3 cms. to 10 cms., with increase of the range from 1·6 to 8·5, which represents over a hundredfold range of the quantity $S=c^3$, the ratio of the resisting forces to the weight. As these two quantities generally vary together, a much greater range in the actual flexural rigidity is covered. The other methods to be described may therefore be regarded as reserves in resource for dealing with special kinds of materials, and not as essential to the stiffness test on the standard instrument, which can deal with a much greater range of materials than, for instance, any one tensile tester.

The first and simplest modification is to deal with very stiff material, such as starched and ironed linen. For this purpose a weight is hung from the free end, using 8 or 9 cm. overhang in order not to foul the quadrant.

The weight may conveniently be a lead hook hung on a loop formed by a thread fastened to each side of the strip of cloth. It may be 0·25 gm. or more according to the material, and there is virtually no limit to the degree of stiffness that can thus be measured. The deflection is observed as usual, and c calculated by the formula given in Appendix A.

There are materials so flimsy that "stiffness" is hardly the word to use in connection with them, and others in which the balance is very unequal, so that the weight of the warp, say, overwhelms any resistance of the sparse weft. Tensile or thickness and compression tests may often be more useful for comparing such materials, but a small increase in the practicable range of measurement by the flexometer may be obtained by observing the deflection of a triangular strip, which gives the same deflection as an equally long rectangle of material with three times the value of S $(=c^3)$. A template, 1 in. wide, was made in the shape of an isosceles triangle, 5 cm. from apex to base, joined by a rectangle, 3 cm. long, to another triangle, 7 cm. from apex to base. Any length of overhang from 3 cm. may be used, and c calculated as described in Appendix A.

The most common difficulty is that presented by the tendency to curl. This may be so pronounced that the strip takes a complete twist, the effect of which on the rigidity cannot be ignored. To inhibit the twist, a long specimen of 20 cm. is cut, the middle marked with a dot, and the two ends placed together to form a pear-shaped loop. The doubled end is placed on the plane D and covered with the weight F, and the deflection measured as before. The depression of the mid-point is much the same as for a strip of the same length as the loop, the length being measured from the nip to the mid-point. The length thus measured, or calculated as equal to 0·4243 times the circumference, may be used to calculate the stiffness (Appendix A).

A material too stiff to be bent thus, but curling badly when tested as a weighted rectangle, may be better dealt with as a triangle weighted at the tip, Equation 6, Appendix A, being used.

The curling is not so pronounced in a broad strip, and an instrument of new design has been made to test strips 6 in. wide. The theory and formulæ are unaffected by this increase in width, and further experience may lead to the adoption of the greater width for general purposes. It has the disadvantage of needing more material, if adequate sampling is to be obtained, but this is counterbalanced by the fact that the warp and weft directions can be tested on the same specimen, 6 in. square. The new form of instrument is illustrated in Fig. 1b.

With the broader instrument, it is also possible to test a specimen cut in circular form, with a segment overhanging (as in Fig. 1b), Equation 3, Appendix A, being used. On this form, measurements can be taken in any direction, and the curl is less pronounced than in any other form.

Really soft fabrics, such as fine hosiery, bend almost to a right angle when overhanging an edge by no more than a centimetre or two. To make their resistance to bending measurable, the amount of bending may be increased. When a strip, 10 cm. long, or less if necessary, is bent into the form of a heart, the ends are turned through an angle of 540°. A perfectly flexible material would fall into a vertical line, but the softest material to be found maintains a very decided loop, the length of which gives a quite satisfactory measure of c. This method has the further advantage of securing the two ends, practically inhibiting curl, while allowing full freedom to bend in the direction desired.

Cloth, if sufficiently stiff and elastic, may be tested by the Searle's pendulum method already described for starch films,[6] whereby the flexural rigidity is determined from the period of vibration. This dynamical method is more laborious, and few cloths are elastic enough to maintain a vibration; it is therefore put forward not as a routine method, but for special research purposes. The decrease of elastic resistance with time affects the measurement much less than in the static test, and the comparison may be used to estimate the extent of the time effect. On suitable materials, it also allows more accurate measurement of the effect of humidity. An example of the latter use is given below.

In all these methods the formulæ developed reduce the observations to the same measure of stiffness, a known function of the forces produced in the bent material, and of its weight. Several experiments have been made to test the identity of c as measured by the different methods, on fabrics, papers, and steel strip. As an example, the following results were obtained on specimens cut from a small piece of mercerised "tarantulle," tested in a room maintained at 70% R.H. and 70° F. The weight of the material was 8·67 mgm. per sq. cm., and its thickness 211 μ. The observations were reduced to values of c by the formulæ given in the Appendix.

Table V

(1) Rectangles			(2) Same as (1), Weighted			(3) Pears			
l	θ	c	l	θ	c	$0{\cdot}424L$	l	θ	c
10	31·6	5·81	9	44·8	5·51	9·34	9·39	28·4	5·77
9	25·0	5·75	8	38·9	5·43	8·49	8·49	22·1	5·74
8	18·8	5·72	7	31·8	5·41	7·64	7·68	16·6	5·75
7	13·0	5·75	6	24·5	5·38	6·79	6·76	11·5	5·73
6	8·4	5·70	5	17·0	5·39	5·94	5·96	7·6	5·86

(4) Triangles			(5) Same as (4), Weighted						(6) Dynamical Test
			7 cm. end			5 cm. end			
l	θ	c	l	θ	c	l	θ	c	
7	3·0	6·52	7	27·1	5·75				Mean period $T=1{\cdot}234$
			6	23·8	5·71				$c=6{\cdot}25$
5	1·7	5·57	5	20·8	5·61	5	14·9	5·79	
			4	16·9	5·61	4	12·0	5·71	
			3	12·9	5·60	3	9·0	5·70	

(1) Standard methods, six specimens. The variations with length are within experimental error and were not consistent in repeat experiments. (2) Weight on end 0·0857 gm., same specimens as (1). The low values may be partly due to handling and were not encountered in other tests. (3) Three specimens, bent into loops of circumference 22 cm., 20 cm., etc. By reversing and turning inside out, four readings are given by each specimen. The measured length of the unstrained loop, l, is not significantly different from the theoretical. (4) Triangles, 7 cm. and 5 cm. long at either end of four specimens. The deflections are too small for accuracy. (5) Same specimens weighted at end with 0·0824 gm. (6) Dynamical test. Four specimens, 5 × 1·5 cm. between grips.

Such comparisons are limited by cloth variability, small changes of condition and after-effects of handling, observational error, and the fact that no one cloth is equally suitable for all methods. There may also be small but real variations of flexural rigidity with the degree of bending, owing to change in shape and in time effects. Several systematic checks were made to test the validity of the formulæ, including those of the circular and the loop methods, and no consistent differences were found in the results of the static methods, but it is advisable, in close comparisons, to keep the method unchanged if possible. The dynamical test gives higher results according to the degree of elastic imperfection, the difference being inappreciable on steel strip (feeler gauge). The ratio of S ($=c^3$) for a series of starch films as obtained dynamically and by the standard method with lengths of 8, 7·5, 7, and 6·5 cms. was 113:100:99:99:101.

Effect of Humidity on the Stiffness of Organdie

It is more convenient to study the effect of humidity by the dynamical than the static method. On a suitable cloth, it allows more accurate measurement of small differences, it introduces less danger of after-effects in a series of tests on the same specimen, and the apparatus is more easily enclosed. Organdie is eminently suited to the dynamical test, being so elastic, and the effect of humidity is of special interest. This material is exported to moist tropical countries, and sometimes becomes too limp to please the purchasers.

A specimen was mounted on the Searle's pendulum, which was enclosed in a chamber with controlled humidity and temperature, having two crank arms passing through the lid by which the pendulum could be set in oscillation. The cloth was first dried for a month over phosphorus pentoxide, which was then replaced by saturated solutions of various salts to give a series of increasing humidities. Readings were taken each day for about a week at each stage, but for three weeks at 97% R.H. and seven weeks at saturation, to see if any steady deterioration was caused by long exposure to moisture. None was noted, for the reading after one day was the same as after seven weeks at saturation. On increasing the humidity, the stiffness fell to a minimum in the first day, then slowly rose to a steady value. On decreasing the humidity, the stiffness rose with decreasing speed to a steady value in about a week.

Table VI

Flexural Rigidity of Organdie at Different Humidities and 20° C. (expressed as ratio of $1/T^2$ to value when dry)

Length between grips, 49·6 mm., width 13·7 mm. Thickness (small light foot), before 92μ, after 101μ.

Date	Control	R.H.	G (absorption)	Desorption	
				G	$G \times (\frac{92}{101})^3$
2/4/27 to 5/5	Phosphorus pentoxide ...	0	1·00	1·31	0·99
	Saturated solutions of—				
12/5	Magnesium chloride ...	34	0·82	—	—
17/5	Sodium dichromate ...	52	0·72	—	—
24/5	Ammonium nitrate ...	65	0·68	0·86	0·65
8/6	Sodium chloride ...	76	0·60	—	—
20/6	Potassium chloride ...	86	0·51	—	—
14/7	Potassium sulphate ...	97	0·27	—	—
30/8	Water	100	0·22	0·22	0·17

The relation for absorption is shown graphically in Fig. 7 and is similar in form and degree to those obtained for the elastic properties of hairs. Up to 80% R.H. the flexural rigidity falls by 5·4% of the dry value for a rise of 10% R.H.

After drying, the thickness was found to have increased by 10%, a change that fully accounted for the increased stiffness after the experiment. Moisture in itself therefore does not appear to destroy the stiff finish permanently, but only while the moisture is retained. The point was important, so nine long specimens were cut along warp and weft, measured by the pear method, and divided into three lots. The first was kept in a chamber over phosphorus pentoxide for three months, the second at 65% R.H., and the third over water. They were then conditioned together for three weeks at 70% R.H. and again tested. Only the last showed an appreciable change, an increased stiffness due to an increased thickness.

Table VII
Permanent Effects of Humidity

	0% R.H.		65% R.H.		100% R.H.		
	Warp	Weft	Warp	Weft	Warp	Weft	
c before exposure	5·16	3·47	5·16	3·47	5·16	3·47	cms.
c after exposure	5·13	3·55	5·16	3·63	5·28	3·95	cms.
Thickness (μ) after exposure	100		101		104		

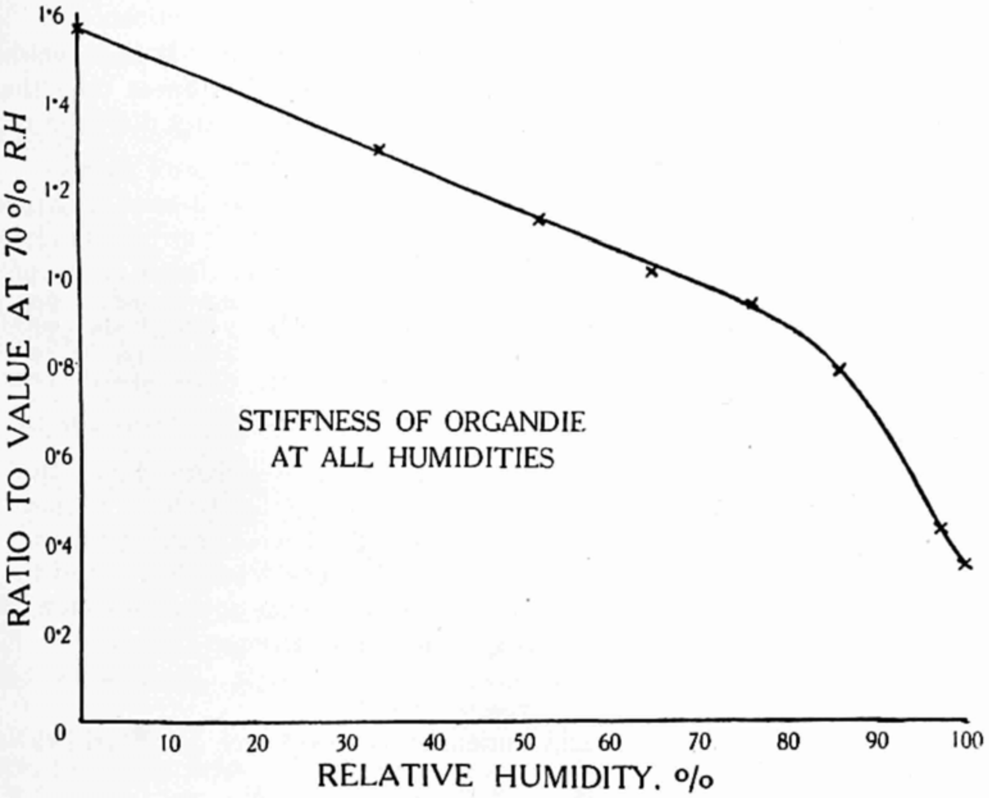


FIG 7

APPENDIX A—REDUCTION TABLES

Rectangular Cantilever

This is the standard method, suitable for most fabrics, and to be used whenever practicable. The length overhanging is adjusted to a suitable value, l, and the angular deflection of the end, θ, is observed, when the bending length

$$c=l.f_1(\theta), \text{ where } f_1(\theta)=(\cos 0{\cdot}5\ \theta\ /\ 8\ \tan\ \theta)^{\frac{1}{3}} \quad \ldots\ldots\ldots\ldots\ldots\ldots 1$$

Table of f_1 (θ)

θ =	0	1	2	3	4	5	6	7	8	9
0		1·928	1·530	1·336	1·213	1·126	1·059	1·005	0·961	0·923
10	0·891	0·862	0·836	0·813	0·792	0·773	0·756	0·739	0·724	0·710
20	697	684	672	661	650	640	630	620	611	602
30	594	585	577	569	562	554	547	540	533	526
40	519	513	506	500	493	487	481	475	468	462
50	456	450	444	438	433	427	421	415	409	403
60	397	391	385	379	373	366	360	354	347	341

Weighted Rectangle

For fabrics too stiff for the standard method, a weight W is fixed to the end of a strip of width b and weight w per unit area.

$$c=l.\left(\frac{W}{3wbl}+0{\cdot}13\right)^{\frac{1}{3}}\cdot\left(\frac{\cos 0{\cdot}93\ \theta}{\tan\theta}\right)^{\frac{1}{3}} \qquad \ldots\ldots\ldots\ldots 2$$

The ratio of cos 0·93 θ to cos θ varies so little over the range used in this method that the value at 20° may be assumed in order to use the table of $f_2(\theta)$ given below, when

$$c=l.f_2(\theta)\ .\ (0{\cdot}336\ W/wbl+0{\cdot}131)^{\frac{1}{3}} \qquad \ldots\ldots\ldots\ldots 2a$$

Circular Cantilever

This method is applicable only to the larger instrument. It best avoids the difficulty of curling, goes rather lower in range of stiffness than the standard method, and allows measurement of stiffness in any direction on one specimen. If r is the radius of the circular specimen,

$$c=l.f_1(\theta)\ .\ f(l/r) \qquad \ldots\ldots\ldots\ldots 3$$

Table of $f(l/r)$

l/r =	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
$f(l/r)$ =	0·811	0·814	0·817	0·821	0·825	0·829	0·834	0·839	0·844	0·850
l/r =	1·0	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9
$f(l/r)$ =	0·857	0·864	0·872	0·881	0·892	0·905	0·921	0·940	0·967	1·007

Hanging Heart

Very limp fabrics beyond the range of the cantilever method may thus be tested. The two ends of a strip are clamped together to form a length L into a heart-shaped loop. From the grip to the lowest or mid-point, the undistorted length of such a loop l_0 is 0·1337 L. The actual length l of the loop hanging under its own weight is measured with a cathetometer or otherwise, and the stiffness calculated from the difference $d=l-l_0$

$$c=l_0 f_2(\theta) \qquad \ldots\ldots\ldots\ldots 4$$

where $\theta=32{\cdot}85^\circ\ .\ d/l_0$, and $f_2(\theta)=(\cos\ \theta\ /\ \tan\ \theta)^{\frac{1}{3}}$.

Other forms, applications, and dimensions of loops are described below and in Appendix B.

Table of f_2 (θ)

θ =	0	1	2	3	4	5	6	7	8	9
0		3·855	3·059	2·671	2·425	2·250	2·115	2·007	1·917	1·841
10	1·774	1·716	1·663	1·616	1·573	1·533	1·496	1·462	1·430	1·400
20	1·372	1·345	1·319	1·294	1·271	1·248	1·226	1·205	1·186	1·164
30	1·144	1·126	1·107	1·089	1·071	1·054	1·037	1·020	1·003	0·986
40	0·970	0·954	0·938	0·922	0·906	0·891	0·875	0·860	0·845	0·829
50	813	799	784	768	753	738	722	707	692	676
60	661	645	630	614	598	582	566	549	533	516
70	499	482	465	447	429	411	392	373	354	333
80	313	291	269	246	222	197	170	140	107	067

As this method is used only for very soft fabrics, it is practicable to fix on a suitable value of L, say, 15 cms., and to construct a table giving the value of c direct from the measured value of l. This will save labour of calculation when the method is used in a routine way.

These four methods are recommended as best for the types of fabric referred to, which include all types amenable to test. The following methods have also been worked out and occasionally applied; they extend the resources of the test when applied to materials other than fabrics.

Triangular Cantilever

In this form the range of the cantilever method is extended a little below that covered by the standard method.

$$c = l.\ 0{\cdot}6933\, f_1(\theta) \qquad \text{.......... 5}$$

Weighted Triangle

Material that is both stiff and curly may be so tested.

$$c = l.\ f_2(\theta).\left(\frac{W}{2wbl} + 0{\cdot}044\right)^{\frac{1}{3}} \qquad \text{.......... 6}$$

Pear-loop Cantilever

This method may be used for soft, curly material when the smaller instrument only is available.

$$c = l.\ 0{\cdot}212/\tan^{\frac{1}{3}}\theta. \qquad \text{.......... 7}$$

Hanging Loops

Various forms of loop may be used to measure the stiffness of very soft materials, and they have the further advantage of minimising the effect of curl or twist, by the positive grip on both ends. While the heart shape seems the most useful, others may have special applications. The ring shape gives most promise of a practicable method for yarns. Using the same symbols as for the heart shape, l_0 is $0{\cdot}3183\ L$, θ is $157{\cdot}0^\circ\ .\ d/l_0$ and

$$c = L.\ 0{\cdot}133\, f_2(\theta) \qquad \text{.......... 8}$$

The last expression holds also for the heart-shaped loop, though the value of the coefficient is a best-fit, not determinable within 1 per cent.

In the hanging pear loop, l_0 is $0{\cdot}4243\ L$, θ is $504{\cdot}5^\circ.\ d/l_0$ and the bending length is given approximately by

$$c = L.\ 0{\cdot}133\, f_2(\theta)\ /\ \cos 0{\cdot}87\,\theta \qquad \text{.......... 8a}$$

Stiffness in any Direction

After measuring the value of c in the warp and weft directions, c_1 and c_2, the value in any other direction at an angle α to the warp is given by

$$c = c_1(\cos^2\alpha + k^2\sin^2\alpha)^{-\frac{1}{3}},\ \text{where}\ k = (c_1/c_2)^{\frac{3}{2}} \qquad \text{.......... 9}$$

and the mean value

$$\bar{c} = \sqrt{c_1.c_2.}\ \tfrac{1}{2}(k + 1/k) \qquad \text{.......... 10}$$

$$= \sqrt{c_1.c_2}\ \text{approximately.}$$

Flexural Rigidity

The weight of the specimen is obtained in milligrams and divided by the area of the template in square centimetres, giving the value of w. When c is expressed in centimetres

$$G = w.c^3 \qquad \text{.......... 11}$$

in mgm. cm., which are convenient units

Bending Modulus

The thickness is measured under the conditions described on p. T388 and conveniently expressed in microns. Then

$$q = 12\ G/d^3 .\ 10^6, \text{ in kgm./cm}^3. \qquad \ldots\ldots 12$$

APPENDIX B—THE MATHEMATICAL BASIS OF THE STIFFNESS TEST

Owing to the comparative softness and variability of cloth specimens, the very small deflections used in elastic measurements on metals are unsuitable, and the mathematical analysis of infinitesimal strains is insufficient. To obtain the relation between large deflections and flexural rigidity, the first step is to formulate the intrinsic differential equation. In general, this is of a form that cannot be integrated as a known function, and various

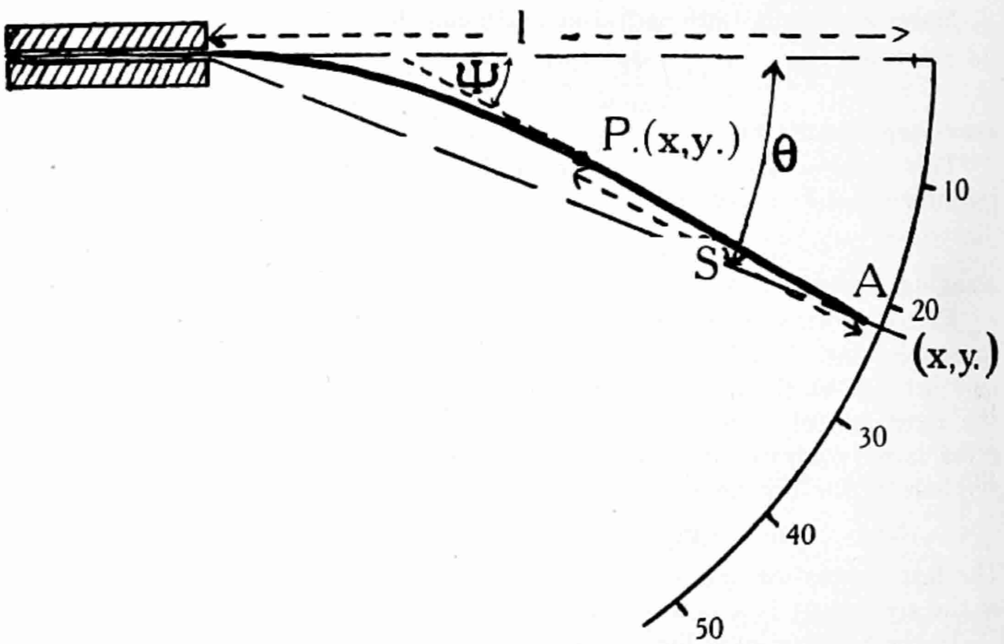


FIG. 8

devices have been used to obtain an expression applicable to experimental data (*a*) by expressing the relation in terms of a known tabulated integral, such as elliptic functions, (*b*) by obtaining a series expansion, and (*c*) by modifying the relation for infinitesimal bending so that it remains true for large bending. The last method has been adopted for this work, so that each relation is expressed in a trigonometrical form in which the factors are determined by methods (*a*) and (*b*), by dimensional considerations, and empirically by observations on feeler steel strip.

The specimen is assumed to be a uniform thin lamina, in which the curvature is proportional to the bending moment.* The flexural rigidity, G, is the bending moment for unit curvature per unit width of strip; the weight per unit area is denoted by w. The immediate object of the test is to evaluate the ratio $S=G/w$, in terms of the overhanging length of the

*This assumption may be emphasised as the basis of the following analysis, in which no consideration is taken of the actual deformations in plates of finite thickness of homogeneous material. Thus the modulus q is descriptive of the material as a whole but bears no known relation to the local stress-strain relations throughout the fabric.

specimen, l, and the angular deflection from the horizontal, θ, of the end of the strip. The bending length, c, is then $\sqrt[3]{S}$.

(1) Rectangle Bending under its own Weight*

This is the standard and simplest test, which is shown diagrammatically in Fig. 8. For infinitesimal bending, the usual expression obtained is

$$S = l^4/8\delta \qquad \text{......} 1a$$

where δ is the depression of the free end. As the deflection increases, this formula gives too high a value for S, since the moment or leverage of the weight is diminished by the bending. If the relation is obtained in terms of θ, infinitesimal bending still being assumed, the formula becomes

$$S = l^3/8 \tan \theta, \qquad \text{......} 1b$$

which remains a good approximation over a greater range, but still needs correction for large deflections. The simplest means is by a factor reducing to unity for $\theta = 0$ and decreasing with θ at a rate increasing with θ: for which a suitable form appears to be $\cos m\theta$, where m is a fraction to be determined.

If s is the distance of a point P along the strip from the free end, ψ the angle between the tangent there and the horizontal (see Fig. 8), the curvature is $d\psi/ds$ and the bending moment $-G.d\psi/ds$. When s changes by a small amount, δs, the change in bending moment, $\delta(-G.d\psi/ds)$, is equal to the change in the moment of the overhanging weight, $ws. \cos \psi. \delta s$, which gives immediately the second differential intrinsic equation of the strip:

$$d^2\psi/ds^2 = -s. \cos \psi/S. \qquad \text{......} 1c$$

This expression applies to many forms of "elastica" traced by strips under their own weight, a hoop being another instance. It is not integrable as a known function. An attempt was made to expand it as a power series, but the few terms obtained diminished so slowly that recourse was had to the empirical method when the stiffness test was first worked out. Hummel and Morton,[1] however, performed the laborious computation of a series of values that put our formula on a more satisfactory basis. A plot of their figures shows that for satisfactory accuracy more terms are necessary in the expansion, but a smooth curve is drawn through the more consistent later values (see below) by the formula adopted here:

$$S = l^3. \cos 0{\cdot}5\, \theta/8 \tan \theta \qquad \text{......} 1$$

These authors use a different practical approximation: in our notation, that

$$S = 32d^3/\, 81\psi^4_0 \qquad \text{......} 1d$$

where ψ_0 is the end slope: also that $S = d^3$ when $\psi_0 = 45°$. The end slope is, however, quite unsuitable for the measurement of the bending of fabric, with its curls and kinks. Nor is the rigorous series expansion, so far as it

*Those whose interest is mathematical may be warned that the analysis which follows is not an exercise in pure mathematics, but is intended to solve in a convenient manner the problems of testing fabric. It is sufficient for its purpose but no claim is made to new advances in the rigorous mathematical theory of elasticity. Rigour does not seem possible in taking account of the weight of the specimen, but where this is lacking, the results have been checked empirically. The approach indicated on p. T409 by dimensional equations with constants determined empirically, would be the least objectionable, but with the material and resources available it was neither accurate nor convenient enough to suffice by itself. A referee has pointed out that the relations for loaded beams might have been deduced at less length from the treatment given in paragraph 97 of Greenhill's *Elliptic Functions*.

has been evaluated, as satisfactory as our trigonometrical approximation, as may be seen from the following comparison of the ratios to $l^3/8 \tan\theta$ of the values of S obtained by the two methods. These ratios are the factor of correction to the formula for infinitesimal bending, and it will be seen that the values obtained by the series are still insufficient approximations, showing an upward trend at the beginning.

θ =	0°	3° 47′	7° 31′	11° 16′	15° 4′	18° 53′	22° 38′	26° 31′	30° 25′	34° 20′
Series	1·0	1·010	1·001	0·996	0·985	0·981	0·965	0·953	0·937	0·920
Formula	1·0	0·999	0·996	0·991	0·985	0·976	0·966	0·954	0·940	0·924

Peterson and Dantzig[4] have more recently given another treatment of this case. Deriving the same series expansion, they use only the first two terms, and transform it into a polar equation.

$$S = r^3 . \frac{\cos 4/3\ \theta}{8\ \theta} \qquad \text{......... 1e}$$

Evidently this cannot be as accurate as the more complete evaluation of the series given by Hummel and Morton, but comparative figures cannot be given without working out the relation between r and l.

(2) Rectangle Weighted at end

This case is also figured by Fig. 8, if the weight W is shown acting at the end point A. This was solved, before the treatment of Hummel and Morton was seen, in a manner similar to that used by Lamb[2] for capillary curves.

The differential equation of the strip is

$$Gb.d\psi/ds = -W(X-x) \qquad \text{......... 2a}$$

where b is the width. Substituting $-dx/\cos\psi = ds$ and integrating,

$$(1 - x/X)^2 = 1 - 2Gb/WX^2.\sin\psi$$

or $\quad v^2 = 2B^2(\cos\beta - \cos\psi') = 4B^2(\cos^2\beta/2 - \cos^2\psi'/2)$

where $\quad B^2 = Gb/WX^2,\ \cos\beta = WX^2/2Gb,\ \psi' = \psi - \pi/2,\ v = 1 - x/X$

Therefore $\quad v = 2Bk.\cos\phi \qquad \text{......... 2b}$

where $\quad k = \cos\beta/2,\ \cos\psi'/2 = k\sin\phi.$

Also $\quad du/d\phi = du/dv.dv/d\phi$, where $u = y/X - C_1$, C_1 a constant.

$$= -\cot\psi'.2Bk\sin\phi = -B\cos\psi'/\sin\psi'/2$$

$$= B\{2\Delta(k,\phi) - 1/\Delta(k,\phi)\}$$

where $\quad \Delta(k,\phi) = \sqrt{1 - k^2\sin^2\phi} = \sin\psi'/2.$

Hence, in the ordinary symbolism of elliptic functions,

$$u = 2B.E(k,\phi) - B.F(k,\phi) \qquad \text{......... 2c}$$

$$v = 2k.B.\cos\phi.$$

To express Gb/W in terms of l and $\tan\theta$ (or Y/X), we take arbitrary values of $\sin^{-1}k$ and calculate the following quantities in order:

$$B^2 = 1/(4k^2-2),\ \phi_0 = \sin^{-1} 1/\sqrt{2k},$$

$$l/X = B\{F(k,\pi/2) - F(k,\phi_0)\},$$

$$-Y/X = l/X - 2B\{E(k,\pi/2) - E(k,\phi_0)\},$$

and $\quad Gb/Wl^2 = B^2/(l/X)^2.$

Plotting the last two quantities against each other gives the required relation.

The familiar relation for infinitesimal bending can be expressed in the form—

$$Gb = Wl^2/3 \tan \theta \qquad \ldots\ldots 2d$$

A comparison of this equation with that derived above results in the application of a correcting factor, cos 0·93 θ, to Equation 2d. This provides a practically exact representation of the rigorous relation over the whole range. The values obtained for Gb/Wl^2 by the three formulæ, (a) the usual form for infinitesimal bending, (b) the corrected formula adopted, (c) the rigorous analysis, are as follows—

θ	0	6° 40′	13° 03′	20° 09′	34° 14′	49° 40′	58° 53′	90°
(a)	0	2·872	1·483	0·979	0·613	0·456	0·422	0·333
(b)	0	2·835	1·406	0·860	0·416	0·196	0·116	0
(c)	0	2·834	1·408	0·860	0·417	0·197	0·115	0

When the bending due to the weight of the strip itself is neglected, the formula adopted is therefore

$$S = \frac{W}{wb} \cdot \frac{l^2 \cos 0{\cdot}93\, \theta}{3 \tan \theta} \qquad \ldots\ldots 2e$$

To allow for the weight of the strip, Equation 1 may be expressed in the form 2d if W is replaced by $\frac{3wbl}{8} \cdot \frac{\cos 0{\cdot}5\, \theta}{\cos 0{\cdot}93\, \theta}$, which roughly represents the effective addition to the end load. For our present purposes, the varying term in θ may be replaced by its value at 20°, and the formula for the combined effect is

$$S = l^3 \left(\frac{\cos 0{\cdot}93\, \theta}{\tan \theta}\right) \left(\frac{W}{3wbl} + 0{\cdot}13\right) \qquad \ldots\ldots 2$$

(3) Triangle Bending under its own Weight

The equation of moments in this case gives

$$\frac{d}{ds}\left(s\,\frac{d\psi}{ds}\right) = s\,\frac{d^2\psi}{ds^2} + \frac{d\psi}{ds} = -\,\frac{s^2}{2S}\cos\psi \qquad \ldots\ldots 3a$$

By the usual approximations, the equation for infinitesimal strain is easily obtained as

$$S = l^3/\,24 \tan \theta. \qquad \ldots\ldots 3b$$

The unsimplified differential equation for large bending can only be integrated by series expansion. Let $t = s^3/\,S$, then equation 3a reduces immediately to the form—

$$d/dt\ (t.d\psi/dt) = -\cos\psi,$$

the solution of which as an intrinsic equation has been obtained in the form—

$$\psi = a - t.\cos a - t^2.\sin 2a/8 + t^3(5\cos a + 3\cos 3a)/144 + t^4(9\sin 4a + 16\sin 2a)/2304 + \qquad \ldots\ldots 3c$$

From the first four terms a relation has to be found between l^3/S and $\tan\theta$. Much labour is saved by doing this for several values of a, the end value of ψ, sufficient to determine the correction to the simple formula 3b.

With a fixed value of a, say 30°, a table is computed of ψ against t. Sin ψ and cos ψ are then plotted against $t^{\frac{1}{3}}$ and integrated by area from t=0 to ψ=0. The ratio of the two areas is the value of tan θ for the given value of end slope. The corresponding value of t_0 is found by substituting $\psi=0$ in Equation 3c, and the reciprocal of this value equals S/l^3. In this way, the following values were found—

For $a=30°$, tan θ 0·4178, $\theta=22° 40'$, $S/l^3=0·0973$, cos $m\theta=0·9755$
45° 0·6881 34° 32′ 0·0579 0·9560

To obtain our usual trigonometrical approximation, S/l^3 is multiplied by 24 tan θ to find the value of the correcting factor, cos $m\theta$, as given above—hence the value of m. In the two examples given above, m was 0·56 and 0·49, and the round figure 0·5 gives S in terms of l^3 and tan θ to within 0·5% (the values of cos 0·5 θ / 24 tan θ being 0·0978 and 0·0578). The formula adopted is therefore

$$S = l^3/3 \,.\, \cos 0{\cdot}5\,\theta \,/\, 8 \tan\theta \qquad \text{.......} 3$$

The same table suffices for this case as for case 1. It is legitimate to assume that the large bending of a uniform unweighted lamina of any shape, at least of those intermediate between a triangle and a rectangle, may similarly be described by the equation for infinitesimal bending modified by the correcting factor cos $\theta/2$. The equation of a cantilever of any shape—rectangle, semi-circle, semi-ellipse, trapezium, triangle—would be obtained by multiplying the right hand side of Equation 1 by a numerical factor calculated for infinitesimal bending.

For a material of known stiffness, the rigorous theoretical deflection of a triangle of given length can be found from the series 3c. Substitute l^3/S for t, put $\psi=0$, and evaluate a by successive approximation: then proceed as above to find tan θ.

(4) Triangle Weighted at end

The bending moment at any point, $G.b/l.s.\ d\psi/ds = -Wx$. Differentiating, we have

$$\frac{s.d^2\psi}{ds^2} + \frac{d\psi}{ds} = -\frac{Wl}{Gb}\,.\cos\psi \qquad \text{.......} 4a$$

This must be integrated by series and the terms converge very slowly.

For infinitesimal strains, the radius of curvature is constant and equal to Gb/Wl, where b is the base, l the length of the triangle. From this the equation is obtained

$$G/W = l^2/2b.\ \tan\theta.$$

From the distribution of curvature, it was judged that the value of m in the correcting term should be if anything greater than for loaded rectangles (0·93), and there can be little error in taking it as unity. Allowing for the weight of the strip, as in case 2 the tentative formula is obtained—

$$S = l^3.\frac{\cos\theta}{\tan\theta}\left(\frac{W}{2wbl} + 0{\cdot}044\right) \qquad \text{.......} 4$$

The case has possible utility in testing materials that are stiff as well as curly.

(5) Segment of Circle Bending under its own Weight

This case is applicable only to the new broad form of tester, but is of special value for specimens that tend to curl; it also has the advantage that the stiffness in any direction can be measured on the one specimen. The symbols of Fig. 8 still apply. In addition, let r be the radius of the circle,

z the length of the semi-chord through P, ω the angle subtended by it. The bending moment at P is $-G.2z.d\psi/ds$, and the differential equation

$$d/ds.-2Gz.d\psi/ds = w.\cos\psi.r^2(\omega - \sin\omega.\cos\omega) \quad \text{..........5a}$$

This reduces to the form

$$d^2\psi/d\omega^2 = -\frac{r^3}{2S}.\sin\omega(\omega-\sin\omega.\cos\omega).\cos\psi \quad \text{..........5b}$$

A first attempt at series expansion did not promise well, so recourse was had to the conclusions arrived at from case 3. Assuming infinitesimal strain, the bending moment at P due to the overhanging weight is

$$w.\int_A^P \text{A}.dx \text{ or } w\int_A^P (x - \text{OP}).d\text{A}.$$

where x is the distance along OA and A the area beyond that distance. Either form integrates easily to give the equation:

$$G.2z.d^2y/dx^2 = -wr^3(-\omega\cos\omega + \sin\omega - 1/3\sin^3\omega) \quad \text{........5c}$$

By the use of the relation $dx=r.\sin\omega.d\omega$, this equation is integrated twice without difficulty, giving the equations for a segment overhanging from a chord that subtends the angle 2Ω.

$$2S.dy/dx = -r^3\left[C_1 - \omega\sin\omega - \frac{5}{3}\cos\omega - \frac{1}{9}\cos^3\omega\right],$$

where $$C' = \Omega\sin\Omega + \frac{5}{3}\cos\Omega + \frac{1}{9}\cos^3\Omega$$

and $$2S.y = r^4\left[C_1\cos\omega + \frac{\omega^2}{4} - \frac{\omega}{2}\cos\omega\sin\omega + \frac{10}{9} - \frac{13}{12}\cos^2\omega - \frac{1}{36}\cos^4\omega\right]_\Omega^\omega \quad \text{5e}$$

In laborious but obvious steps, this leads to the expression

$$S = f(\Omega).l^3/8\tan\theta \quad \text{..........5f}$$

where $$f(\Omega) = \left(40 - 2\cos\Omega + \cos^2\Omega - 3\cos^3\Omega + 9\Omega.\frac{\Omega - \sin 2\Omega}{1-\cos\Omega}\right)\Big/ 9(1-\cos\Omega)^3$$

and $\cos\Omega = 1 - l/r$, l being as usual the length overhanging.

As a segment is intermediate in form between a rectangle and a triangle, it is legitimate to assume the same correcting factor that proved suitable in both these cases for transforming the equation of infinitesimal to that of finite bending. Thus—

$$S = f(\Omega).l^3\cos 0{\cdot}5\,\theta/8\tan\theta \quad \text{..........5}$$

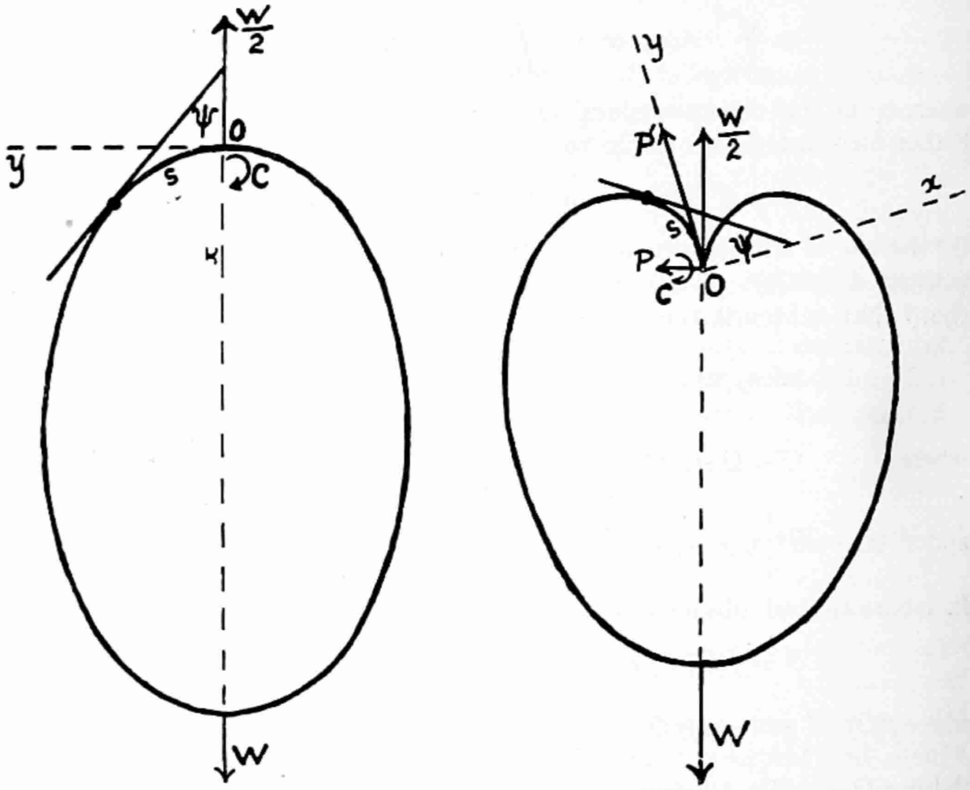
For a semi-circle, the value of $f(\Omega)$ is 0·62866. To reduce observations to the quantity c, a table is given on p. T400 of the values of $f^{\frac{1}{3}}(\Omega)$ as $f(l/r)$.

(6) Bending "Pear"

The two ends of a strip are brought together and secured between the platform and weight of the stiffness tester, and the pear-shaped loop so formed is allowed to overhang. The differential equation for this case could not be integrated, by series or otherwise, so there is no purpose in reproducing it. The necessary equation was obtained by a purely empirical method. Loops of feeler steel, of various lengths and thicknesses, were observed in this form, and the deflection of the mid-point was compared with that observed by the simple cantilever method. It was found that the function $L^3/\tan\theta$ remained sensibly constant, and that the stiffness was given by the formula

$$S = l^3_0/8\tan\theta,$$

where l_0, the undistorted length of the loop, is equal to 0·4243 L, the circumference of the loop, from the analysis given on p. T414. The relation has been checked on paper and fabrics, and is sufficiently accurate and well established for the purposes of practical measurement.



(*a*) Ring (*b*) Heart

FIG. 9

Hanging loops

The loops are drawn to the shape calculated for $k = \sin 80°$.

(7) Hanging Loops

Materials with a bending length of less than some 2 cms. are too soft to be tested satisfactorily by any cantilever method, in which the maximum amount of bending is 90°. By enforcing a much greater bending, more resistance may be developed, and its effects may be made measurable even with specimens of the softest knitted fabric. If the two ends of a weightless, elastic strip are held in grips closer together than the length of the strip, it takes a form in which the curvature at any point is proportional to the moment of the end constraints, and which is called an "elastica." A strip of perfectly flexible, heavy material, a chain, similarly constrained, takes another form in which the tangent of the angle that the tangent of the curve makes with the horizontal is proportional to the arc measured from the lowest point, and which is called a "catenary." An actual strip of any material, which has in general both flexural rigidity and weight, takes an intermediate form from which the ratio of these two quantities can be deduced.

If the two ends of the strip are brought into contact, a loop is formed, the length of which, when allowed to hang straight downwards from the grip, is a sensitive measure of the ratio, G/w. Three loops of this kind have been studied, where one end has been brought against the other by bending through angles of 180° (pear), 360° (ring), and 540° (heart) (Fig. 9). Let l_0 be the length, from grip to lowest point, of the elastica of circumference L and d the depression caused by the weight. Then it may be seen from dimensional considerations that the value of d depends only on $S=G/w$ and L, and that the relation should be expressible in the form—

$$S = L^3 / f(d/L) \qquad \text{.......7a}$$

The value of d/L is limited between 0 for $S=\infty$ to the value given by a catenary, $D/L=0{\cdot}5-l_{\cdot}/L$ when $S=0$. A suitable manner of expressing the strain is therefore in terms of $\tan\theta$ where $\theta=\frac{\pi}{2}\cdot\frac{d}{D}$. The relations for the several loops have been evaluated by analysis and by observations on feeler steel. Rigorous analysis of the form of a ring distorted by its own weight proved very intractable. Some rigour, but apparently no significant accuracy was sacrificed by solving the problem in the following manner. Relations were obtained for the length of a loop, of negligible weight, distorted by a weight W hung from the lowest point. Both for the ring and heart, these can be very satisfactorily expressed by the simple formula—

$$G/W = k.\, L^3.\, \cos\theta \,/\, \tan\theta \qquad \text{.......7b}$$

Equilibrium means a state of minimum potential energy, so that a minute displacement involves no change of energy, i.e. the drop of the weight represents the same amount of energy as the additional strain in the strip. Now, in a ring distorted by a weight, the centre of gravity of the strip is half the length of the loop below the grip, and moves half as far as the weight when an additional distortion is imposed. The effect of the weight of the strip may therefore be allowed for by an addition of $wL/2$ to the hanging weight W; or, if no weight is suspended, by substituting $wL/2$ for W, giving the relation—

$$S = G/w = k.\, L^3 \cos\theta \,/\, \tan\theta \qquad \text{.......7}$$

The lack of rigour in this derivation lies in the fact that the form of a ring distorted by its own weight is somewhat different from that produced by a concentrated added weight. The latter is symmetrical about the horizontal axis, the former has a greater curvature at the top. On this account the centre of gravity moves rather more than half the change in length of the loop. On the other hand, the strain energy increases rather more for the same length change, so that the equivalence of $wL/2$ to W should not be greatly affected. The empirical check serves to confirm this.

The same device was used to derive the equation for a heart-shaped loop. Taken over the full range from elastica to catenary, the centre of gravity moves at a rate 0·652 of the change in length, and W is replaced by 0·652 wL. In Fig. 10 the analytical relations are compared with those observed for strips of feeler steel, of various length and thicknesses. The latter are subject to the errors of measurement of both the loop and cantilever methods, including variations from uniformity and straightness in the strips, and deviate as much from each other as from the analytical relation. The value thus obtained for k in Equation 7 is not appreciably different from that

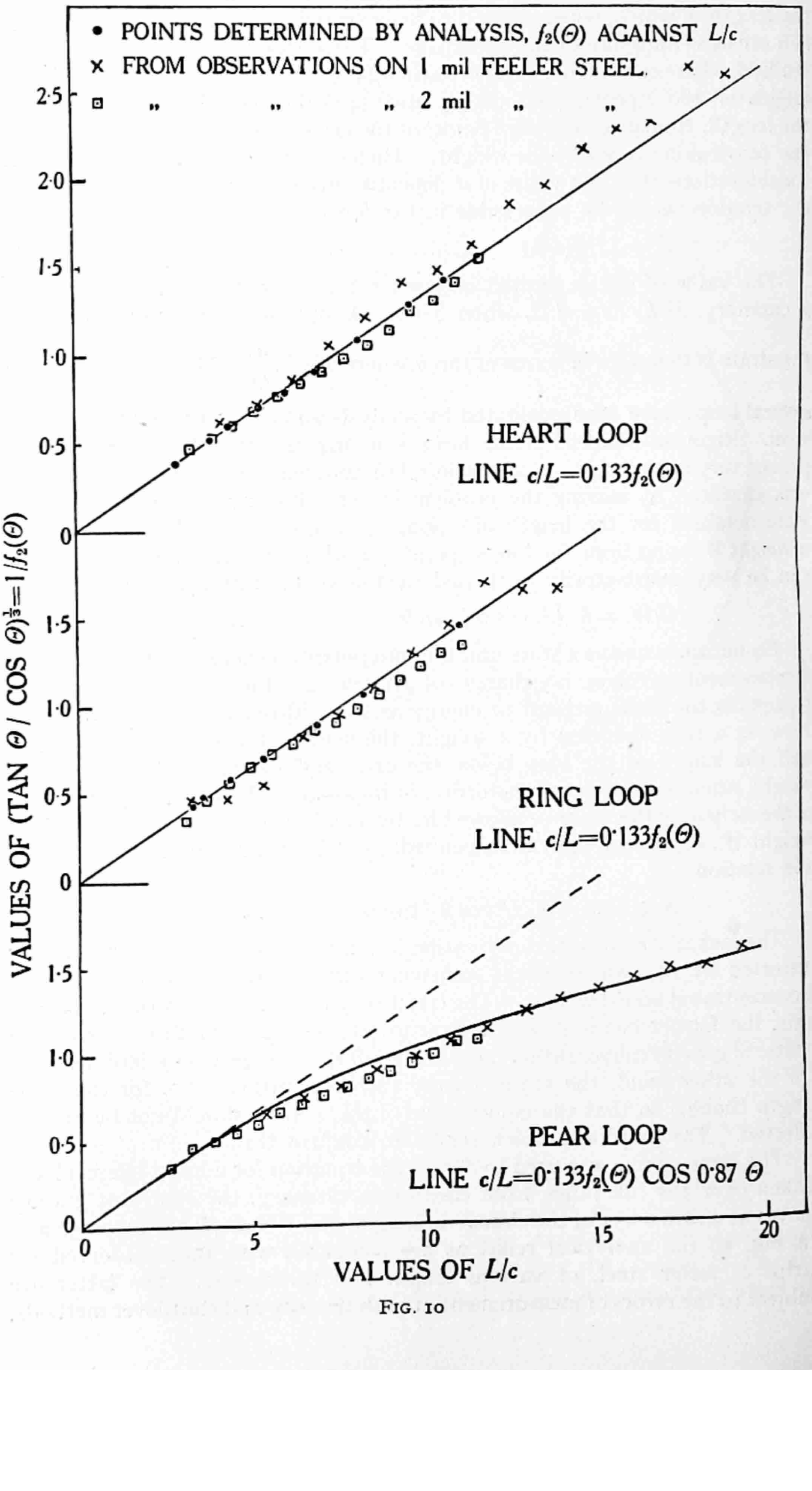


FIG. 10

obtained analytically, and may be fixed at 0·133. Equation 7 is then accurate within the accuracy obtainable with feeler steel, which gives more reproducible and regular results than any other material observed.

As far as present experience goes, the heart loop is of most value as a means of extending measurement of stiffness to the softest fabrics, while the ring offers the most promise for the measurement of yarn stiffness. The pear loop is less sensitive, and has not been so fully analysed, but the empirical relation has been observed for feeler steel and is also shown in Fig. 10. Over a range of small distortions it would appear that the same relation held as for the other two forms, but when L/c exceeds 4, the value of θ falls below that given by Equation 7. In this form of loop the weight of the strip is much less effective than a suspended weight, particularly in bending the bottom of the loop when it approaches the catenary form. The observations on the pear loop indicate that a similar deviation from Equation 7 may be expected of the other loops at extreme values of L/c. If so, they are fortunately inappreciable over the range of practical utility, for the observations of the heart loop follow the relation to the highest value of L/c observed, which is nearly 20. It is quite possible to form a loop of soft material with 5 cms. or less, when this value of L/c corresponds to a bending length of 0·25 cm., much less than any value yet observed.

Throughout the mathematical analysis it has been assumed that the bending moment is proportional to the curvature. The question of time effects must be considered separately, but even when Hooke's law is obeyed the ratio of moment to curvature increases when the latter becomes excessive. Such considerations, however, need hardly concern us in the measurement of quality.

(8) Analysis of the Ring Loop

The symbols shown in Fig. 9 being used, the equation of moments for a ring distorted by its own weight gives—

$$\delta . G \frac{d\psi}{ds} = -\delta\, C + \delta \frac{wL}{2}\, y - \delta s .\, \sin \psi . ws$$

which can be reduced to the form—

$$B \frac{d^2\psi'}{du^2} = -\, u \cos \psi' \qquad \ldots\ldots 8a$$

where $\psi' = \psi + \pi/2,\ u = 1 - 2S/L$, and $B = 8G/wL^2$.

From this equation ψ' was expressed in a series of powers of u, but not in a form sufficiently convergent to give reasonable accuracy.

The intrinsic differential equation of a ring distorted by a suspended weight W is—

$$-\, G \frac{d\psi}{ds} = C - \frac{W}{2}\; y \qquad \ldots\ldots 8b$$

Substituting $dy/\sin\, \psi$ for ds and integrating

$$(c - y)^2 = c^2 - 2b^2 \cos \psi = \frac{4b^2}{k^2}\,(1 - k^2 \sin^2 \varepsilon)$$

where $c = 2C/W,\ b^2 = 2G/W,\ k = 2b/\sqrt{c^2 + 2b^2}$, and $\varepsilon = \pi/2 - \psi/2$,

or $$y = \frac{2b}{k}\left(\sqrt{1 - k^2/2} - \sqrt{1 - k^2 \sin^2 \varepsilon}\right) \qquad \ldots\ldots 8c$$

Also $\dfrac{dx}{d\varepsilon} = \cot\psi \, . \, \dfrac{dy}{d\varepsilon} = -\dfrac{2b}{k} \, . \, \dfrac{k^2 \cos 2\varepsilon}{2\sqrt{1-k^2\sin^2\varepsilon}}$

and $$x = \frac{2b}{k}\left[(1-k^2/2)\,(F-F_0) - (E-E_0)\right] \qquad \text{.....................8d}$$

where $F=F(k.\varepsilon)$, $F_0=F(k, \pi/4)$, etc., standard elliptic functions.

Also $-\, b^2 \, . \, \dfrac{d\psi}{ds} = (c-y) = \dfrac{2b}{k}\sqrt{1-k^2\sin^2\varepsilon}$

and $$s = bk(F-F_0) \qquad \text{...8e}$$

When the end values $x=X$, $\varepsilon=\dfrac{3\pi}{4}$, and $s=L/2$ are substituted in 8d and 8e, the relation between the elongation of the ring and G/W is obtained in the form—

$$\frac{b}{L} = \frac{1}{4k(F_1-F_0)}$$

$$\frac{X-X_0}{L} = \frac{(F_1-F_0)-(E_1-E_0)}{k^2(F_1-F_0)} - 0{\cdot}81831 \qquad \text{..............8f}$$

For the relation between elongation and G/w in a ring bending under its own weight, $wL/2$ is substituted for W, when $c/L=(b/2L)^{1/3}$. The extreme limits of X are from L/π to $L/2$. and the elongation is expressed in terms of $\theta=90°(X-X_0)/0{\cdot}18169\,L$. A series of values of k being substituted, the following points were determined—

$\sin^{-1}k$	45°	50°	60°	70°	80°	85°	89°
L/c	3·232	3·571	4·341	5·332	6·851	8·210	10·988
θ	4·790	6·351	10·845	18·076	31·031	42·237	58·228

These points are plotted in Fig. 10 as $(\tan\theta/\cos\theta)^{1/3}$ against L/c, and compared with observations on feeler steel of 1 and 2 mil thickness. For these, θ is the value observed for rings of circumference L, c is derived from observations by the standard cantilever method.

Two problems are solved by this analysis, though the second does not concern the present subject. The stiffness of a ring distorted by its own weight is given by—

$$c/L = 0{\cdot}133\,(\cos\theta/\tan\theta)^{1/3},$$

and, when a ring is used as a spring or as a structural member, the relation between its elongation and load is given by —

$$\tan\theta/W\cos\theta = 0{\cdot}0047\,L^2/G. \qquad \text{.............................8g}$$

NOTE—The rigorous equation 8a can be solved in the form—
$U=\pi u+B(\pi u+\pi u.\cos\pi u-2\sin\pi u+A_1u+A_5u^5+A_7u^7+ \;.\;.\;.\;.\;.\;)$
where $A_1+A_5+A_7+\;.\;.\;.\;.\;=0$. The series in u proves awkward. When it is ignored, a moderate approximation is given for slight distortion, but apparently not as good as that adopted.

(9) Analysis of Heart Loop

In a heart loop from which a weight W is suspended, the constraints at the grip on the left end of the strip, Fig. 9, may be expressed as a pressure upwards of $W/2$, horizontally leftwards of P and a clock-wise couple or bending moment C. The ordinate y' is measured parallel to the resultant, P', of the pressures, x' being measured upwards at right angles to y'.

The intrinsic differential equation of the loop is—

$$G\frac{d\psi'}{ds} = P'x' + C \qquad \text{...9a}$$

Substituting $dx/\cos\psi'$ for ds, and integrating

$$(x'+c)^2 = c^2 - 2b^2\sin\alpha + 2b^2\sin\psi'$$
$$= 4\frac{b^2}{k^2}(1-k^2\sin^2\varepsilon),$$

where $b^2 = G/P'$, $c = C/P'$, $\alpha = \psi'_0$, the angle between P' and P, $2\varepsilon = \pi/2 - \psi'$, and $k = 2b/\sqrt{c^2+2b^2(1-\sin\alpha)}$.

or
$$x' = \frac{2b}{k}\left(\sqrt{1-k^2\sin^2\varepsilon} - \sqrt{1-k^2\sin^2\varepsilon_0}\right) = \frac{2b}{k}(\Delta - \Delta_0) \qquad \text{......9b}$$

Also
$$\frac{dy'}{d\varepsilon} = \tan\psi'\frac{dx'}{d\varepsilon} = -\frac{2b}{k}\cdot\frac{k^2\cos 2\varepsilon}{2\Delta}$$

and
$$y' = \frac{2b}{k}\left[(1-k^2/2)(F-F_0) - (E-E_0)\right] \qquad \text{..................9c}$$

Also
$$b^2\frac{d\psi'}{ds} = x' + c = \frac{2b}{k}\cdot\Delta$$

and
$$s = bk(F_0 - F) \qquad \text{..................9d}$$

At the bottom of the loop $\psi = \dfrac{3\pi}{2} + \alpha$, and for convenience we shall write

$$\delta F = F\left(k, \frac{\pi}{4} - \frac{\alpha}{2}\right) - F\left(k, -\frac{\pi}{2} - \frac{\alpha}{2}\right),\ \delta E = E\left(k, \frac{\pi}{4} - \frac{\alpha}{2}\right) - E\left(k, -\frac{\pi}{2} - \frac{\alpha}{2}\right)$$
$$\text{and } \delta\Delta = \Delta - \Delta_0.$$

Substituting the value $s = L/2$, we obtain $2b/L = 1/(k\delta F)$, and the value $y = 0 = -x'\sin\alpha + y'\cos\alpha$ gives the relation between k and α as $\tan\alpha.\delta\Delta = (1-k^2/2)\delta F - \delta E$. From this, pairs of values must be obtained by successive approximations. Eight such pairs of values were determined, and a smooth curve was drawn giving the value of k for any value of α from 0 to 90°.

Substituting the end values in the equation, $-x = x'\cos\alpha + y'\sin\alpha$, and using the above expression for $\tan\alpha$,

$$\frac{X}{L} = \frac{\delta\Delta}{k^2\delta F\cos\alpha} \qquad \text{..................9e}$$

Returning to the original value of b and α, $2G/W = b^2/\sin\alpha$,

or
$$WL^2/8G = k^2.\delta F^2.\sin\alpha \qquad \text{..................9f}$$

These two equations, 9e and 9f, define the relation for a weighted loop. For a strip under its own weight alone, we must find the weight W that is equivalent to the distributed weight wL, by comparing the movements of the lowest point and the centre of gravity. The vertical position of the latter is given by

$$\bar{x} = \frac{2}{L}\int x\frac{bk}{\Delta}\,d\varepsilon \qquad \text{..................9g}$$

It is possible by graphical methods to compare the two rates of movement at all values of α, but the simpler approximation has been adopted of comparing the total movements to get an average value for wL in terms of W. Equation 9g is easily integrated for the undistorted loop to give $\bar{x}/L = 0{\cdot}0112$, which increases to 0·25 L for the catenary, while the lowest point moves from 0·1337 L to 0·5 L, whence $W = 0{\cdot}652\ wL$, and

$$L/c = 2{\cdot}306\ (k^2\delta F^2\sin\alpha)^{\frac{1}{3}} \qquad \text{..................9h}$$

The results of the calculations from the above formulæ are shown in the following table and plotted in Fig. 10. They are again well expressed by the formula—

$$c/L = 0{\cdot}133\ (\cos\theta\ /\ \tan\theta)^{\frac{1}{3}}$$

As the constant is not significantly different from 0·1337, this may also be written: $c = l\ (\cos\theta/\tan\theta)^{\frac{1}{3}}$, though no theoretical significance is attached to this simplification. The observations on feeler steel are in excellent agreement with the analysis, as far as available tables allow the results of the latter to be calculated, and continue to follow the above formula to the greatest distortion observed, where θ is 77°.

Table VIII

α	$\sin^{-1} k$	L/c	X/L	θ
0°	66·5°	0	0·1337	0
12	65	2·826	1472	3·32°
20	64·3	—	—	—
30	64	3·848	1673	8·25
41·7	65	4·374	1832	12·16
45	65·7	—	—	—
57·3	70	5·260	2090	18·49
66·7	75	6·024	2320	24·19
73·7	80	6·896	2584	30·63
80·0	85	8·169	3026	41·50
83·6	88	9·643	3482	52·69
84·94	89	10·670	3685	57·68
90	90	∞	0·5000	90

(10) Analysis of Pear Loop

This form of loop may be analysed in the same manner as the heart loop, but it is unnecessary here to pursue the analysis so far, as the hanging pear does not seem to offer any special advantages. The loop has been used as a cantilever, and it also offers a convenient method for measuring the limit of flexibility, so the form of the undistorted loop will be demonstrated here.

The end restraints may be expressed as an inward pressure P and a couple C. Measuring y along the axis, x the half-width, and ψ the angle between the tangent and the normal to the axis—

$$G\frac{d\psi}{ds} = Py - C \qquad \text{10a}$$

and $(y - c)^2 = c^2 - 2b^2\cos\psi$,

where $c = C/P$ and $b^2 = G/P$. Substituting $\cos\beta = c^2/2b^2$ and $\cos\psi/2 = \cos\beta/2.\ \sin\phi = k\sin\phi$, this reduces to

$$y = 2kb(\cos\phi - \cos\phi_0) \qquad \text{10b}$$

Also $\dfrac{dx}{d\phi} = \cot\psi\,\dfrac{dy}{d\phi} = -\dfrac{b\cos\psi}{\sin\psi/2}$

and $x = b(2E - F)$ ... 10c

Also $b^2 d\psi/ds = y - c = 2kb\cos\phi$ ... 10d

whence $s = b(F_0 - F)$. ... 10e

When y is a maximum at the end of the loop, ϕ is zero, giving from 10*e*, $2b/L = 1/F_0$, and from 10*c*, $F_0 = F(k, \phi_0) = 2E_0$. From the last condition, the value of k is determined as 0·8554 = sin 58·8°, and ϕ_0 as 124·25°. Then b/L is 0·1587, and y (max.), the length of the loop, is 0·4243 L.

From the above equations the form of the loop was plotted for a circumference L of 20 inches, and the form was found identical with that of a loop of feeler steel.

(11) A Test for the Limit of Flexibility

From equation 10d, the radius of curvature at any point

$$R = ds/d\psi = L/10{\cdot}77 \cos \phi, \qquad \text{.................................11a}$$

and the radius of curvature at the end of the loop is 0·093 L, the curvature being greatest at this point. This fact offers a convenient method for observing the effect of excessive bending on moderately brittle materials.

The test has been applied to starch film to measure the maximum strain when the specimen snaps. A strip is bent into pear shape, and the nip is moved forward to make the pear smaller and smaller till it snaps, when the circumference L is given by the position of the nip. The strain is very localised, and is a maximum in the outer skin at the end. To compare materials of different thicknesses, the strain may therefore be calculated as

$$\text{Maximum strain} = d/2R = 5{\cdot}385\, d/L \qquad \text{.....................11}$$

Among a series of starch films of varying composition (additions of waxes, etc.), the flexibility varied in rough agreement with the ultimate extensibility, but was consistently much higher. In the tensile test the extension is cut short by rupture of the weakest flaw, and the ultimate value observed is far below the real limit at which the material ruptures.

The same test may be used to test the cracking of varnished (electrical) or doped (aircraft) fabric, and may possibly be applied to stiff filled and calendered material.

(12) Stiffness in any Direction

A formula is given by Love[3] for the Young's modulus in any direction of a body with three planes of symmetry at right angles. Cloth may be treated as a lamina with two planes of symmetry and the Young's modulus in any direction in the plane, for which the third direction cosine is always zero, is given by —

$$1/E = l^4/E_1 + m^4/E_2 + 2l^2m^2/F_3.$$

E_1, E_2 and F_3 are given in terms of the co-ordinates of the strains in a very general strain energy-function but experimentally it appears that $F_3^2 = E_1 . E_2$ and

$$1/E = (l^2/\sqrt{E_1} + m^2/\sqrt{E_2})^2$$

or, as w and d are not dependent on the direction,

$$1/S = (\cos^2 \alpha + k^2 \sin^2 \alpha)^2/S_a,$$

where α is the angle with the warp, S the value of S in the warp direction, $k^4 = S_a/S_e$.

The mean value of S in any direction $\overline{S}$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} S\,.\,d\alpha$$

$$= \frac{2Sa}{\pi}\int_0^{\frac{\pi}{2}} \frac{(1+t^2).dt}{(k^2+t^2)^2} \quad \text{where } t = \tan\alpha$$

$$= \frac{2Sa}{\pi}\left[\frac{1}{k}\,.\,\frac{k^2+1}{2k^2}\,.\,\tan^{-1}\frac{t}{k} + \frac{1-k^2}{2k^2}\,.\,\frac{t}{t^2+k^2}\right]_{\theta=0}^{\theta=\frac{\pi}{2}}$$

reducing to $\overline{S} = \sqrt{S_a\,.\,S_e}\,.\,\frac{1}{2}\,(1/k + k)$.

The formulæ may be used also for G or for q as for S, and the formulæ for c are given immediately by substituting c^3 for S. They are, of course, not rigorous, for the "Young's Modulus" in different directions of a fabric is a quantity scarcely capable of exact definition, but the forms suggested by the theory of homogeneous material are found to be satisfactory empirically.

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